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When is Cartesian product a Cayley graph?

(Extended abstract)

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Abstract

A graph is said to be *Cayley* graph if its automorphism group admits a regular subgroup. Automorphisms of the Cartesian product of graphs are well understood, and it is known that Cartesian product of Cayley graphs is a Cayley graph. It is natural to ask the reverse question, namely whether all the factors of Cartesian product that is a Cayley graph have to be Cayley graphs. The main purpose of this paper is to initiate the study of this question.

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1 Introduction

Throughout this paper graphs are assumed to be finite, simple, and connected, and groups are finite. Given a graph Γ we let $V(\Gamma)$, $E(\Gamma)$, and $\mathrm{Aut}(\Gamma)$ be the set of vertices, the set of edges, and the automorphism group of Γ, respectively.

Let G be a finite group and $S \subseteq G \setminus \{1\}$ an inverse closed subset of G. Then the *Cayley graph* $Cay(G, S)$ *on* G *with respect to* S is a graph with vertex set G and edge set $\{\{g, gs\} \mid g \in G, s \in S\}$. It is well-known that a graph Γ is a Cayley graph on a group G if there exists a regular subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to G (see [5]).

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Recall that the *Cartesian product* $\Gamma_1 \Box \cdots \Box \Gamma_k$ of graphs $\Gamma_1, \ldots, \Gamma_k$ has vertex set $V(\Gamma_1) \times \cdots \times V(\Gamma_k)$ with two distinct vertices being adjacent if they are adjacent in one of the coordinates and coincide in all other coordinates. Recall also that two graphs are called *relatively prime* if there exists no non-trivial graph that is a factor – with respect to the Cartesian product – of both of them. A graph is said to be *prime* with respect to the Cartesian product if it cannot be factored as a Cartesian product of two non-trivial graphs. For a graph Γ, the Cartesian product $\underbrace{\Gamma \Box \ldots \Box \Gamma}_{\text{n times}}$ is denoted with $\Gamma^{\Box n}$.

It is well-known that the Cartesian product of Cayley graphs is a Cayley graph. A natural question is to consider whether the converse is true, that is, if the Cartesian product of graphs is a Cayley graph, does each of the factors have to be a Cayley graph? This is the main motivation for the work presented in this article. We provide partial results showing that Cartesian products involving certain vertex-transitive non-Cayley graphs are not Cayley (for example, every graph having a Petersen graph as one of the factors is non-Cayley). We are not aware of any example of a Cayley graph having a non-Cayley factor.

2 Preliminaries

We start by recalling the structure of the automorphism group of the Cartesian products.

Theorem 2.1. *[2, Theorem 6.8] Let Γ be a connected graph with prime factorization $\Gamma = \Gamma_1 \Box \Gamma_2 \Box \ldots \Box \Gamma_k$. Then for any automorphism φ of Γ, there is a permutation π of $\{1, 2, \ldots, k\}$ and isomorphisms $\varphi_i : \Gamma_{\pi(i)} \to \Gamma_i$ for which*

$$\varphi(x_1, x_2, \ldots, x_k) = (\varphi_1(x_{\pi(1)}), \ldots, \varphi_k(x_{\pi(k)})).$$

Let $G_i \leq Sym(V_i)$ for $i \in \{1, \ldots, n\}$. The group $G_1 \times G_2 \times \ldots \times G_n$ acts *canonically* on $V_1 \times V_2 \times \ldots \times V_n$ in such a way that $g_i \in G_i$ is applied to the i-th coordinate. We have the following simple observation.

Lemma 2.2. *Let $G_i \leq Sym(V_i)$ for $i \in \{1, \ldots, n\}$ be transitive groups. If there exists a regular subgroup of $G_1 \times G_2 \ldots \times G_n$ acting canonically on $V_1 \times V_2 \times \ldots \times V_n$, then every G_i admits a regular subgroup.*

Proof. Let H be a regular subgroup of $G_1 \times G_2 \ldots \times G_n$. Since H is regular, it follows that $|H| = |V_1| \cdot \ldots \cdot |V_n|$. Let $j \in \{1, \ldots, n\}$, and let $v_i \in V_i$ be arbitrary for $i \neq j$. Let $K = \{(g_1, \ldots, g_n) \in H \mid g_i(v_i) = v_i \; i \in \{1, \ldots, n\} \setminus \{j\}\}$ and $K(j) = \{g_j \mid (g_1, \ldots, g_j, \ldots, g_n) \in K\}$. It is easy to see that $K(j)$ is a subgroup of G_j, and that the transitivity of H implies that $K(j)$ is transitive subgroup of G_j. Moreover, if $K(j)$ is not semiregular, then H would contain a non-identity element fixing a point of $V_1 \times \ldots \times V_n$, contrary to the assumption that H is regular. We conclude that $K(j)$ is a regular subgroup of G_j. Since j is arbitrary, the result follows. $\square$

The following result follows directly by applying Lemma 2.2 to the fact that the automorphism group of Cartesian product of relatively prime graphs is the direct product of the automorphism groups of the factors, see [2, Corollary 6.12] (see also [3, Theorem 3.1]).

Theorem 2.3. *Let Γ_1 and Γ_2 be two connected relatively prime graphs with respect to the Cartesian product and let $\Gamma = \Gamma_1 \Box \Gamma_2$. Then Γ is a Cayley graph if and only if both Γ_1 and Γ_2 are Cayley graphs.*

In light of Theorem 2.3, the question of which Cartesian products are Cayley is reduced to the question when is a Cartesian power of a graph isomorphic to a Cayley graph. In the following result the automorphism group of a Cartesian power of a graph is given. Let us first recall the definition of a wreath product of permutation groups. Let $G \leq Sym(V)$ and $H \leq S_n$. The *wreath product of* G *by* H denoted by $G \wr H$ is the set of all permutations $((g_1, \ldots, g_n), h)$ of V^n (where $g_1, \ldots, g_n \in G$ and $h \in H$) such that $((g_1, \ldots, g_n), h) : (v_1, \ldots, v_n) \mapsto (g_{h(1)}(v_{h(1)}, \ldots, g_{h(n)}(v_{h(n)})$.

Lemma 2.4. *Let Γ be a graph that is prime with respect to the Cartesian product. Then* $\mathrm{Aut}(\Gamma^{\Box n}) \cong \mathrm{Aut}(\Gamma) \wr S_n$.

3 Main results

The following result giving a bound on the order of a Sylow p-subgroup of the symmetric group S_n will be needed later.

Lemma 3.1. *Let $n \geq 1$ be an integer and p a prime divisor of n. A Sylow p-subgroup of S_n has order less than p^n.*

Proof. Let $n = a_0 + a_1 p + \ldots + a_k p^k$ with $0 \leq a_i \leq p-1$. By [4, pg. 11] a Sylow p-subgroup of S_n has order p^M, where

$$\begin{aligned} M &= \sum_{i=1}^{k} a_i (1 + p + p^2 + \ldots + p^{i-1}) = \sum_{i=1}^{k} a_i \frac{p^i - 1}{p-1} \\ &= \sum_{i=1}^{k} \left[\frac{a_i p^i}{p-1} - \frac{a_i}{p-1} \right] = \frac{\sum_{i=1}^{k} a_i p^i}{p-1} - \sum_{i=1}^{k} \frac{a_i}{p-1} \\ &\leq \frac{n}{p-1} - \sum_{i=1}^{k} \frac{a_i}{p-1} < \frac{n}{p-1} \leq n. \end{aligned}$$

□

An automorphism of a graph is said to be *semiregular* if all the cycles in its cyclic decomposition have equal lengths.

Theorem 3.2. *Let Γ be a vertex-transitive graph such that* $\mathrm{Aut}(\Gamma)$ *has no semiregular element of order p for some prime p dividing $|V(\Gamma)|$, and Γ is prime with respect to the Cartesian product. Then $\Gamma^{\square n}$ is not Cayley for every $n \geq 1$.*

Proof. We first observe that Γ is not Cayley, as if it were, it would contain a regular subgroup R of order n. Then R has a subgroup of prime order p, which is necessarily semiregular, contradicting out assumption.

Suppose that R is a regular subgroup of $\mathrm{Aut}(\Gamma^{\square n})$. As Γ is prime with respect to the Cartesian product, by Lemma 2.4, we have that $\mathrm{Aut}(\Gamma^{\square n}) = \mathrm{Aut}(\Gamma) \wr S_n$ with the product action. Let P be a Sylow p-subgroup of R. Observe that P has order at least p^n. Define $\varphi : P \to S_n$ with $\varphi((g_1, \ldots, g_n), \sigma) = \sigma$ (where $g_i \in \mathrm{Aut}(\Gamma)$ and $\sigma \in S_n$). Observe that φ is a homomorphism. If the kernel of φ is trivial, then P is isomorphic to a subgroup of S_n. However, by Lemma 3.1, the order of a Sylow p-subgroup of S_n is less than p^n. It follows that the kernel of φ is not trivial. It follows that there exists a non-identity element $\gamma = ((g_1, \ldots, g_n), id) \in P$. Without loss of generality we may assume that the order of γ is p (by taking the p-th powers of γ if necessary), implying that each g_i is identity or of order p. Since by the assumption, $\mathrm{Aut}(\Gamma)$ has no semiregular element of order p, it follows that each g_i of order p fixes a vertex of Γ. Hence γ fixes some point in $V(\Gamma)^n$. But as R is regular, and $\gamma \in P \leq R$, this means $\gamma = 1$, a contradiction. □

Corollary 3.3. *No Cartesian power of the Petersen graph is isomorphic to a Cayley graph.*

Proof. Let P denote the Petersen graph. By Theorem 3.2, we need only show that $\mathrm{Aut}(P)$ has no semiregular element of order 2. The automorphism group of the Petersen graph is isomorphic to the action of S_5 on the 2-subsets of $\{1, 2, 3, 4, 5\}$ by [1, Theorem 2.1.4]. The elements of order 2 in S_5 are a product of two transposition as well as transpositions. It is easy to see that the 2-subset of $\{1, 2, 3, 4, 5\}$ which is permuted in a transposition, is fixed by a transposition, and so no element of order 2 in the automorphism group of the Petersen graph is semiregular. □

Theorem 3.4. *Let Γ be a vertex-transitive graph that is not isomorphic to a Cayley graph, whose automorphism group has order relatively prime to $n!$. Then $\Gamma^{\square n}$ is not isomorphic to a Cayley graph.*

Proof. Suppose that R is a regular subgroup of $\mathrm{Aut}(\Gamma^{\square n})$. Then R has order relatively prime to $n!$, in which case every element of R must fix every factor of $V(\Gamma)^n$ (i.e. no element of R can permute factors of $V(\Gamma)^n$). This means that $R \leq \mathrm{Aut}(\Gamma)^n$, hence the result follows by Lemma 2.2. □

Corollary 3.5. *Let Γ be a vertex-transitive graph of odd order that is not a Cayley graph. Then $\Gamma \square \Gamma$ is not isomorphic to a Cayley graph.*

In the following result we study the structure of a transitive permutation group G admitting no regular subgroup, but such that $A \wr S_2$ in the product action admits a regular subgroup.

Theorem 3.6. *Let $A \leq Sym(V)$ be transitive. If $A \wr S_2$ admits a regular subgroup (in the product action) then A admits a regular subgroup or A admits a semiregular subgroup with two orbits.*

Proof. Let $H \leq A \wr S_2$ be a regular subgroup. If $H \leq A \times A$, then by Lemma 2.2 it follows that A admits a regular subgroup. Suppose that H is not contained in $A \times A$. Let $\overline{H} = H \cap (A \times A)$. Observe that $\overline{H}$ is an index two subgroup of H. Since H is regular, it follows that $\overline{H}$ is semiregular with two orbits.

Let $\overline{H}(v) = \{h_1 \in A \mid \exists h_2 \in A_v \text{ such that } (h_1, h_2) \in \overline{H}\}$. Observe that $\overline{H}(v)$ is a subgroup of A. Moreover, it is semiregular, since $\overline{H}$ is semiregular. If $\overline{H}(v)$ is transitive or has two orbits then we are done. Hence, we may assume that $|Orb_{\overline{H}(v)}(x)| \leq |V|/3$ for every $v \in V$.

Let O be one of the two orbits of $\overline{H}$ on $V \times V$. Let $O(v) = \{y \in V \mid (y, v) \in O\}$. Let $x \in O(v)$ be arbitrary. We claim that $O(v) = Orb_{\overline{H}(v)}(x)$. Let $y \in O(v)$. Then (x, v) and (y, v) belong to the same orbit O of $\overline{H}$, hence there exists $(h_1, h_2) \in \overline{H}$ such that $h_1(x) = y$ and $h_2(v) = v$. It follows that h_1 is an element of $\overline{H}(v)$ mapping x to y. This shows that $O(v)$ is contained in $Orb_{\overline{H}(v)}(x)$.

Let z be an element of $Orb_{\overline{H}(v)}(x)$. There exists $h_1 \in \overline{H}(v)$ such that $h_1(x) = z$. By the definition of $\overline{H}$ it follows that there exists $h_2 \in A$ fixing v such that $(h_1, h_2) \in \overline{H}$. This shows that (h_1, h_2) is an element of $\overline{H}$ mapping (x, v) to (z, v), hence (x, v) and (z, v) belong to the orbit O, implying that $z \in O(v)$. This shows that $O(v) = Orb_{\overline{H}(v)}(x)$.

It is easy to see that $O = \bigcup_{v \in V} O(v)$ is a partition of O, and that $|O| = \sum_{v \in V} |O(v)|$. Since $|Orb_{\overline{H}(v)}(x)| \leq |V|/3$, it follows that $|O| \leq |V|^2/3$, contradicting the assumption that $|O| = |V|^2/2$. The obtained contradiction shows that $\overline{H}(v) \leq A$ must be regular or semiregular with two orbits for some $v \in V$. □

Remark 3.7. *There are examples of transitive groups without regular subgroups such that their wreath product with S_2 in the product action admits regular subgroups. For example, TransitiveGroups(24)[675] (of order 288 and degree 24) is one such group. However, the authors are not aware of any such group which is automorphism group of a graph.*

Corollary 3.8. *Let Γ be a graph that is prime with respect to the Cartesian product such that $\mathrm{Aut}(\Gamma)$ admits no semiregular subgroup with two orbits. Then $\Gamma \square \Gamma$ is not isomorphic to a Cayley graph.*

Remark 3.9. *There exist infinitely many vertex-transitive graphs of even order that do not admit a semiregular subgroup with two orbits. One such graph is the Tutte-Coxeter graph, which is a cubic symmetric graph of order 30. Moreover, there exist vertex-transitive graphs of even order admitting a semiregular automorphism of order p, for every prime divisor of the order of the graph, but not admitting a semiregular subgroup with 2 orbits. In particular, any vertex-transitive graph of order 2^n without a semiregular subgroup with 2 orbits is such an example.*

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