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Powers of planar graphs, product structure, and blocking partitions

(Extended abstract)

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Abstract

We show that there exist a constant c and a function f such that the k-power of a planar graph with maximum degree Δ is isomorphic to a subgraph of $H \boxtimes P \boxtimes K_{f(\Delta,k)}$ for some graph H with treewidth at most c and some path P. This is the first product structure theorem for k-powers of planar graphs, where the treewidth of H does not depend on k. We actually prove a stronger result, which implies an analogous product structure theorem for other graph classes, including k-planar graphs (of arbitrary degree).

Our proof uses a new concept of blocking partitions which is of independent interest. An ℓ-blocking partition of a graph G is a partition of the vertex set of G into connected subsets such that every path in G of length greater than ℓ contains two vertices in one set of the partition. The key lemma in our proof states that there exists a positive integer ℓ such that every planar graph of maximum degree Δ has an ℓ-blocking partition with parts of size bounded in terms of Δ.

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1 Introduction

Given two graphs[1] G and H, their *strong product* $G \boxtimes H$ is defined as the graph on $V(G) \times V(H)$ where distinct vertices $(u_1, v_1), (u_2, v_2) \in V(G) \times V(H)$ are adjacent if $u_1 = u_2$ and $v_1v_2 \in E(H)$, or $u_1u_2 \in E(G)$ and $v_1 = v_2$, or $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$.

Graph product structure theory describes complicated graphs as subgraphs of the strong products of graphs with a simple structure like graphs of bounded treewidth[2], paths, or small complete graphs. Arguably the most important result of this theory is the product structure theorem for planar graphs by Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood [7], which has been the key to solving many long-standing problems [1, 2, 4–7, 9, 10]. This theorem states that every planar graph is contained in $H \boxtimes P \boxtimes K_3$ for some graph H with $\text{tw}(H) \leqslant 3$ and some path P. Here a graph G is *contained in* a graph G' if G is isomorphic to a subgraph of G'.

Another product structure for planar graphs by Ueckerdt, Wood, and Yi [14] states that every planar graph is contained in $H \boxtimes P$ for some graph H with $\text{tw}(H) \leqslant 6$ and some path P. Note that $H \boxtimes P$ is isomorphic to $H \boxtimes P \boxtimes K_1$. These two product structure theorems for planar graphs illustrate a trade-off between the treewidth of H and the size of the complete graph involved in the product: If we want to find some fixed planar graph in a graph of the form $H \boxtimes P \boxtimes K_d$ where $\text{tw}(H) \leqslant c$ for some constant c and P is a path, then we can either have $c = 3$ and $d = 3$, or $c = 6$ and $d = 1$.

There are many other graph classes $\mathcal{G}$ for which there exist constants c and d such that every $G \in \mathcal{G}$ is isomorphic to a subgraph of $H \boxtimes P \boxtimes K_d$ for some graph H with $\text{tw}(H) \leqslant c$ and some path P [3, 7, 8, 11, 12]. The strong product $H \boxtimes P \boxtimes K_d$ is isomorphic to $(H \boxtimes K_d) \boxtimes P \boxtimes K_1$ and $\text{tw}(H \boxtimes K_d) \leqslant d(c+1) - 1$, so it is always possible to drive d down to 1, while minimising c is usually more difficult. Moreover, in many applications of such product structure theorems, the main dependency is on c. Therefore, the primary goal is to minimise c, whereas minimising d is a secondary goal. This paper proves new product structure theorems for k-powers of planar graphs of bounded degree and k-planar graphs. The distinguishing feature of our results is that the bound c on $\text{tw}(H)$ is an absolute constant which does not depend on k.

For an integer $k \geqslant 1$, the *k-power* of a graph G is the graph G^k on $V(G)$ where two distinct vertices u and v are adjacent if and only if the distance between u and v in G is at most k. Dujmović et al. [8] proved that for every planar graph G of maximum degree Δ, and for every integer $k \geqslant 1$, the k-power G^k is contained in $J \boxtimes P \boxtimes K_{6k\Delta^k(k^3+3k)}$ for some graph J of treewidth at most $\binom{k+3}{3} - 1$ and some path P. Note that dependence on Δ is unavoidable since, for example, if G is the complete $(\Delta-1)$-ary tree of height k, then G^{2k} is a complete graph on roughly $(\Delta-1)^k$ vertices. Ossona de Mendez [13] asked whether this bound on $\text{tw}(H)$ could be made independent of k. We show that indeed this is the

[1] We consider simple, finite, undirected graphs G with vertex-set $V(G)$ and edge-set $E(G)$.

[2] The *treewidth* $\text{tw}(H)$ of a graph H, is the least integer k such that H is a subgraph of a graph G on a set $\{v_1, \dots, v_n\}$ such that $n \geqslant k+1$ and for each $i \in \{k+1, \dots, n\}$, the neighbours of v_i in $\{v_1, \dots, v_{i-1}\}$ form a clique of size k in G.

case.

Theorem 1. *There exist a constant c and a function f such that for every planar graph G of maximum degree Δ and every integer $k \geqslant 1$, the graph G^k is contained in $H \boxtimes P \boxtimes K_{f(\Delta,k)}$ for some graph H with* $\mathrm{tw}(H) \leqslant c$ *and some path P.*

For an integer $k \geqslant 1$, a *k-planar* graph is a graph which has a drawing on the plane such that no three edges cross at a single point and each edge is involved in at most k crossings. Dujmović et al. [8] proved that every k-planar graph is contained in $H \boxtimes P \boxtimes K_{18k^2+48k+30}$, for some graph H with $\mathrm{tw}(H) \leqslant \binom{k+4}{3} - 1$ and some path P. Dujmović et al. [8] asked whether this bound on $\mathrm{tw}(H)$ could be made independent of k. We give an affirmative answer to this question.

Theorem 2. *There exist a constant c and a function f such that every k-planar graph G is contained in $H \boxtimes P \boxtimes K_{f(k)}$ some graph H with* $\mathrm{tw}(H) \leqslant c$ *and some path P.*

Theorems 1 and 2 follow from Theorem 4, which we formulate in Section 2. In particular, in our proof the value of c is the same in both theorems and equal to 15086399. This value is not optimal, but instead of optimising the constant c we chose to simplify the proof.

The proof of Theorem 4 uses a new concept of "blocking partitions". For an integer $\ell \geqslant 1$, an *ℓ-blocking partition* of a graph G is a partition $\mathcal{R}$ of $V(G)$ such that every set in $\mathcal{R}$ induces a connected subgraph of G and every path of length greater than ℓ in G contains two vertices in one part of $\mathcal{R}$. The *width* of $\mathcal{R}$ is the maximum size of a part of $\mathcal{R}$.

The following lemma plays the key role in our proof.

Lemma 3. *There exists a function f such that every planar graph of maximum degree at most Δ has a 222-blocking partition of width at most $f(\Delta)$.*

The construction of a 222-blocking partition is inspired by chordal partitions of triangulations by van den Heuvel et al. [15]. In their construction, a triangulation G is partitioned into paths $P_1, \ldots, P_m$ where each path P_j is a shortest path between two distinguished vertices in one component of $G - \bigcup_{i<j} V(P_i)$. In our construction, we partition a planar graph G into trees $T_1, \ldots, T_m$ where each T_j is obtained as follows. First, we define T_j^0 as a smallest tree in $G - \bigcup_{i<j} V(T_i)$ which contains some distinguished set of vertices of bounded size, and then, the tree T_j is obtained from T_j^0 by attaching all adjacent vertices as leaves. Finally, we split each tree T_j into subtrees of bounded size by removing an appropriate set of edges. Then, the vertex-sets of these subtrees define the desired 222-blocking partition of G.

Lemma 3 is the most technical part of our proof, and we do not include its proof here. Instead, we sketch the proof of the main theorems assuming Lemma 3. The partition in our proof of Lemma 3, is actually ℓ-blocking for some value ℓ significantly smaller than 222, but we decided to prove a worse bound on ℓ for simplicity's sake.

2 The main result

A *congested model* of a graph G' in a graph G is a set $(B_x : x \in V(G'))$ of connected subgraphs of G such that for every edge $xy \in E(G')$, the subgraphs B_x and B_y *touch* in G, i.e. they share a vertex or there is an edge between $V(B_x)$ and $V(B_y)$ in G. A *rooted congested model* of G' in G is a set $((B_x, v_x) : x \in V(G'))$ such that $(B_x : x \in V(G'))$ is a congested model of G' in G and $v_x \in V(B_x)$ for each $x \in V(G')$. We call a rooted congested model $((B_x, v_x) : x \in V(G'))$ in G an (r, Δ, d)*-model* if

- in each B_x, all vertices are at distance at most r from v_x,
- in each B_x, every vertex distinct from v_x has degree at most Δ, and
- for every $u \in V(G)$, there exist at most d vertices $x \in V(G')$ with $u \in V(B_x)$.

We call a graph G' an (r, Δ, d)*-minor* of G if there exists an (r, Δ, d)-model of G' in G. Note that G' is a minor of G if and only if G' is an $(r, \Delta, 1)$-minor of G for some $r, \Delta \geqslant 0$. A graph G' is an r*-shallow minor* of G if G' is an $(r, \Delta, 1)$-minor of G for some $\Delta \geqslant 0$. Observe that if G' is an (r, Δ, d)-minor of a graph G, then G' is an r-shallow minor of $G \boxtimes K_d$.

If G is a graph with maximum degree at most Δ, then G^k is an (r, Δ, d)-minor of G for $r = \lfloor k/2 \rfloor$ and $d = \sum_{i=0}^{\lfloor k/2 \rfloor} \Delta^i$, as witnessed by the rooted congested model $((B_x, x) : x \in V(G^k))$ where each B_x is the subgraph of G induced by the vertices at distance at most $\lfloor k/2 \rfloor$ from x. Furthermore, it is easy to see that every k-planar graph G' is an (r, Δ, d)-minor of G for $r = \lceil k/2 \rceil$ and $\Delta = d = 2$, where G is the planar graph obtained from G' by adding a dummy vertex at each intersection point. Therefore, Theorems 1 and 2 follow from the following theorem.

Theorem 4. *There exists a function f such that every (r, Δ, d)-minor of a planar graph is contained in $J \boxtimes P \boxtimes K_{f(r,\Delta,d)}$ for some graph J with $\operatorname{tw}(J) \leqslant 15086399$ and some path P.*

Theorem 4 implies a constant-treewidth product structure for other graph classes like δ-string graphs or k-fan-bundle graphs (we refer the reader to [11] for the definitions of these classes).

While it was easy to see that Theorem 4 implies Theorems 1 and 2, it is less obvious why Lemma 3 implies Theorem 4. The main idea behind this implication is captured by the following lemma.

Lemma 5. *There exists a function g such that for any r, Δ, d with $r \geqslant 224$, $\Delta \geqslant 0$ and $d \geqslant 1$, every (r, Δ, d)-minor of a planar graph is an $(r - 1, \Delta', d')$-minor of some planar graph for some $d', \Delta' \in \{1, \dots, g(\Delta, d)\}$.*

Proof. Let f be the function from Lemma 3, and set $g(\Delta, d) = \max\{d, \Delta\} \cdot f(d\Delta)$. Let G be a planar graph, let G' be an (r, Δ, d)-minor of G, and let $((B_x, v_x) : x \in V(G'))$ be an (r, Δ, d)-model of G' in G. Let $G_0 = \bigcup_{x \in V(G')} B_x - v_x$. Note that G_0 is a subgraph of G of maximum degree at most $d\Delta$. Let $\mathcal{R}$ be a 222-blocking partition of G_0 of width at most $f(d\Delta)$, and let us define $\mathcal{R}' = \mathcal{R} \cup \{\{v\} : v \in V(G) \setminus V(G_0)\}$. Let H denote the *quotient* $G/\mathcal{R}'$, i.e., let H be a graph on $\mathcal{R}'$ where two distinct parts are adjacent if G contains an edge with ends in these two parts. Since G is planar and each part of $\mathcal{R}'$ is connected, H is a minor of G, and thus a planar graph. Let $d' = df(d\Delta)$, and let $\Delta' = \Delta f(d\Delta)$. Since the width of $\mathcal{R}$ is at most $f(d\Delta)$, each part of $\mathcal{R}$ has degree at most Δ' in H. Hence, G' is an (r, Δ', d')-minor of H with a corresponding (r, Δ', d')-model $((B'_x, v'_x) : x \in V(G'))$ in H defined as follows. For each $x \in V(H)$, let v'_x be the part of $\mathcal{R}'$ containing v_x, and let B'_x be the "projection" of B_x on G', so that the vertices of B'_x are those parts of $\mathcal{R}'$ which contain at least one vertex of B_x, and two parts are adjacent in B'_x if B_x contains an edge with ends in those parts.

We claim that $((B'_x, v'_x) : x \in V(G'))$ is actually an $(r-1, \Delta', d')$-model. To show that, we need to prove that for any $x \in V(G')$ and $u' \in V(B'_x)$, the distance between v'_x and u' in B'_x is at most $r-1$. Let u be a vertex of B_x which belongs to the part u' of $\mathcal{R}'$. Let $u_0 \cdots u_s$ be a shortest path in B_x with $u_0 = v_x$ and $u_s = u$. Since $((B_x, v_x) : x \in V(G'))$ is an (r, Δ, d)-model, we have $s \leqslant r$. For each $i \in \{0, \ldots, s\}$, let u'_i be the part of $\mathcal{R}'$ containing u_i. Hence, $u'_0 = v'_x$, $u'_s = u'$, and for each $i \in \{0, \ldots, s-1\}$, either $u'_i = u'_{i+1}$ or $u'_i u'_{i+1} \in E(H)$. Therefore, the distance between v'_x and u' is at most s, and thus at most r. Suppose towards a contradiction that this distance is exactly r. Hence, $s = r$, and the vertices $u'_0, \ldots, u'_r$ are pairwise distinct parts of $\mathcal{R}'$. Therefore, $u_1, \ldots, u_r$ is a path in G_0, with no two vertices in one part of $\mathcal{R}$. As $r \geqslant 224$, the length of this path is at least 223, which contradicts $\mathcal{R}$ being 222-blocking. This proves that G' is an $(r-1, \Delta', d')$-minor of H. $\square$

The proof of Theorem 4 uses Lemma 5 and the following result by Hickingbotham and Wood [11].

Theorem 6 ([11])**.** *If a graph G is an r-shallow minor of $H \boxtimes P \boxtimes K_d$ where $\mathrm{tw}(H) \leqslant t$, then G is contained in $J \boxtimes P \boxtimes K_{d(2r+1)^2}$ for some graph J with $\mathrm{tw}(J) \leqslant \binom{2r+1+t}{t} - 1$.*

Proof of Theorem 4. Let g be the function from Lemma 5. We may assume that $g(\Delta, d) \leqslant g(\Delta', d')$ whenever $\Delta \leqslant \Delta'$ and $d \leqslant d'$. Define $f(r, \Delta, d)$ recursively:

$$f(r, \Delta, d) = \begin{cases} 3d(2r+1)^2 & \text{if } r \leqslant 223, \\ f(r-1, g(\Delta, d), g(\Delta, d)) & \text{if } r \geqslant 224. \end{cases}$$

We show that this function satisfies the theorem by induction on r. Let G be a planar graph, and let G' be an (r, Δ, d)-minor of G. For the base case, suppose that $r \leqslant 223$. By the product structure theorem for planar graphs, G is contained in $H \boxtimes P \boxtimes K_3$ for some graph H with $\mathrm{tw}(H) \leqslant 3$ and some path P. Hence, G' is an r-shallow minor of $H \boxtimes P \boxtimes K_{3d}$. By Theorem 6, G' is contained in $J \boxtimes P \boxtimes K_{3d(2r+1)^2}$ for some graph J with $\mathrm{tw}(J) \leqslant \binom{2r+1+3}{3} - 1 \leqslant \binom{450}{3} - 1 = 15086399$.

For the induction step, suppose that $r \geqslant 224$. By Lemma 5, there exist a planar graph H and $d', \Delta' \in \{1, \ldots, g(\Delta, d)\}$ such that G' is an $(r-1, \Delta', d')$-minor of H. By the induction hypothesis, there exists a graph J with $\mathrm{tw}(J) \leqslant 15086399$ such that G' is contained in $J \boxtimes P \boxtimes K_{f(r-1,\Delta',d')}$. Since $d' \leqslant g(\Delta, d)$ and $\Delta' \leqslant g(\Delta, d)$, we have $f(r-1, \Delta', d') \leqslant f(r, \Delta, d)$, and therefore G' is contained in $J \boxtimes P \boxtimes K_{f(r,\Delta,d)}$. This completes the proof. $\square$

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