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Cycles through two edges in signed graphs

(Extended abstract)

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Abstract

We give a characterization of when a signed graph G with a pair of distinguished edges $e_1, e_2 \in E(G)$ has the property that all cycles containing both e_1 and e_2 have the same sign. This answers a question of Zaslavsky.

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1 Introduction

Throughout we assume (signed) graphs to be finite and loopless (loops add nothing to the problem under consideration), but we permit parallel edges. A *signed* graph is a triple $G = (V, E, \sigma)$ where (V, E) is a graph and $\sigma : E \to \{-1, 1\}$ is a *signature*. We say that the *sign* of a cycle $C \subseteq G$ is *positive (negative)* if $\sigma(C) = \prod_{e \in E(C)} \sigma(e)$ is equal to 1 (-1). If all cycles of G are positive, then we call G *balanced* and otherwise we call G *unbalanced*.

In a 2-connected signed graph G, a single edge e appears in cycles of both signs if and only if $G - e$ is unbalanced. For the "only if" direction, let C_1, C_2 be cycles of opposite sign containing e and note that the symmetric difference of $E(C_1)$ and $E(C_2)$ is a set of edges with negative sign and even degree at every vertex (which can thus be expressed as a disjoint union of edge sets of cycles). For the "if" direction, let $e = uv$, choose a negative cycle C in $G - e$, and apply Menger to choose two vertex disjoint paths from $\{u, v\}$ to $V(C)$; these two paths together with C and e contain the desired cycles.

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Our objective in this article is to extend this simple property to a pair of edges. If G is a signed graph and $e_1, e_2 \in E(G)$, then we say that e_1 and e_2 are *untied* if there exist cycles containing e_1 and e_2 of both positive and negative sign, and otherwise we say that e_1 and e_2 are *tied*. Our main result is as follows.

Theorem 1.1. *Let G be a 3-connected signed graph and let $e_1, e_2 \in E(G)$ be distinct and not in parallel with any other edges. Then e_1 and e_2 are tied in G if and only if one of the following holds:*

1. *There exists a parallel class F containing edges of both signs so that $F^+ = F \cup \{e_1, e_2\}$ is an edge-cut and $G - F^+$ is balanced,*

2. *e_1, e_2 are incident with a common vertex v and $G - v$ is balanced,*

3. *$G - \{e_1, e_2\}$ is balanced.*

In Section 2, we provide a reduction that allows us to determine the structure of arbitrary signed graphs that are tied, meaning this result implies a full characterization of when all cycles through two given edges of a signed graph have the same sign. This problem was explicitly asked by Zaslavsky in [13, E2], but let us remark that our motivation for this work is a forthcoming application of these results in the setting of nowhere-zero flows on signed graphs, towards Bouchet's conjecture that every flow-admissible signed graph has a nowhere-zero 6-flow [1]. We apply the results here while finding a decomposition of the edges of a 3-connected signed graph similar to Seymour's decomposition in the first proof of his 6-Flow Theorem [9].

Theorem 1.1 may be viewed as a signed graph generalization of the following result from Lovász's problem book [8, 6.67]. By replacing the edge e_3 of Theorem 1.2 with two parallel edges, one of each sign, forming a signed graph with exactly one negative edge, one observes that Theorem 1.1 does indeed imply Theorem 1.2.

Theorem 1.2. *[Lovász] Let G be a simple 3-connected graph and $e_1, e_2, e_3 \in E(G)$ be distinct. Then there is no cycle containing e_1, e_2, e_3 if and only if one of the following holds:*

1. *$G - \{e_1, e_2, e_3\}$ is disconnected,*

2. *e_1, e_3, e_3 are incident with a common vertex.*

Another generalization of Theorem 1.2 is the following conjecture by Lovász [7] and Woodall [12] (independently): If G is a k-connected graph, and $S \subseteq E(G)$ a set of k independent edges so that either k is even or $G - S$ is connected, then there is a cycle $C \subseteq G$ with $S \subseteq E(C)$. Kawarabayashi [4] showed that S is always contained in either one cycle or two vertex-disjoint cycles. And Thomassen and Häggkvist [3] showed that the conjecture holds if one assumes G is $(k+1)$-connected. The following well-known conjecture of Lovász also concerns connectivity, paths and cycles.

Conjecture 1.3. *[Lovász] For any natural number k, there exists a least natural number $f(k)$ so that for any $f(k)$-connected graph G and any $x, y \in V(G)$ there exists an induced xy-path P so that $G - V(P)$ is k-connected.*

The above conjecture also has a natural generalization to signed graphs that we state below. To deduce 1.3 from 1.4, simply add a single negative edge xy to the graph (treat all other edges as positive).

Conjecture 1.4. *For any natural number k, there exists a least natural number $f'(k)$ so that for any $f'(k)$-connected, unbalanced, signed graph G there exists an induced negative cycle C so that $G - V(C)$ is k-connected.*

Concerning the two conjectures above, Tutte [10] proved the simplest of these cases, that $f(1) = f'(1) = 3$. Using Tutte's language, a cycle C in a graph G is *peripheral* if C is induced and $G - V(C)$ is connected. Tutte showed that every 3-connected graph has a peripheral cycle through any given edge, so $f(1) = 3$. Moreover, he proves that the peripheral cycles generate the cycle space. That is to say that the peripheral cycles are not contained in any codimension 1 subspace of the cycle space. It follows that every signed graph with a non-trivial signature has a negative peripheral cycle, and $f'(1) = 3$. Kriesell [6] and independently Chen Gould and Yu [2] show that $f(2) = 5$.

And so we have provided two examples of interesting statements about graphs which have a natural and more general interpretation in the setting of signed graphs.

2 Outline of the Proof

2.1 Reduction to 3-connected

A *k-separation* of a graph G is a pair of subgraphs (G_1, G_2) so that $E(G_1) \cap E(G_2) = \emptyset$, $E(G_1) \cup E(G_2) = E(G)$, and $|V(G_1) \cap V(G_2)| = k$. We say that the separation is *proper* if $V(G_1) \setminus V(G_2) \neq \emptyset \neq V(G_2) \setminus V(G_1)$.

Observation 2.1. *Let G be a 2-connected signed graph, let $e_1, e_2 \in E(G)$, and let (G_1, G_2) be a 2-separation of G with $V(G_1) \cap V(G_2) = \{u, v\}$. For $i = 1, 2$ let G_i^+ be obtained from G_i by adding a positive edge f_i with ends u, v.*

1. *If $e_i \in E(G_i)$ for $i = 1, 2$ then e_1 and e_2 are tied in G if and only if e_i and f_i are tied in G_i^+ for $i = 1, 2$.*

2. *If $e_1, e_2 \in E(G_1)$ and every edge in $E(G_2)$ is positive, then e_1 and e_2 are tied in G if and only if they are tied in G_1^+.*

3. *If $e_1, e_2 \in E(G_1)$ and G_2 is unbalanced, then e_1 and e_2 are tied in G if and only if they are tied in the graph obtained from G_1^+ by adding a negative edge f_1' in parallel with f_1.*

The above observation allows us to reduce the problem of two edges being tied to one on smaller graphs. Continuing in this manner, we may reduce the problem to the setting of 3-connected signed graphs. Since the concept of edges e_1, e_2 being tied is vacuous if e_1 and e_2 are in separate blocks, Theorem 1.1 gives a complete answer. All of the steps in our reduction are reversible, so we can turn this around and provide a generic construction of signed graphs where two given edges are tied by taking the three types given in the above theorem and combining them as in the observation. The possible structures of all such graphs can readily be determined but we found no better way of describing them than by way of the decompositions presented here.

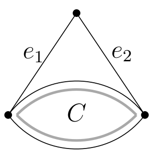
2.2 Some forbidden minors in tied signed graphs

In the setting of signed graphs we are principally focused on the signs of cycles and not those of edges. Accordingly, two signatures σ, σ' of a signed graph G are *equivalent* if every cycle $C \subseteq G$ satisfies $\sigma(C) = \sigma'(C)$. Two signatures are equivalent if and only if one can be obtained from the other by a *switch*, which is changing the sign of every edge in some edge-cut.

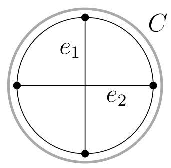
Let $G = (V, E, \sigma)$ be a signed graph and let $e \in E$ ($v \in V$). To *delete* the edge e (vertex v) we remove this edge (vertex and all incident edges) from the graph and adjust the domain of σ accordingly. To *contract* the edge e, first modify σ by switching on an edge-cut (if necessary) so that $\sigma(e) = 1$, and then modify the graph by contracting e and removing e from the domain of σ. If H is a signed graph obtained from G by a (possibly empty) sequence of edge and vertex deletions and edge contractions, we call H a *minor* of G. Note that whenever $C \subseteq H$ is a cycle, there is a corresponding cycle $C^* \subseteq G$ containing all edges in C and having the same sign as C. In particular, this implies the following key property.

Observation 2.2. *Let H be a minor of the signed graph G. If e_1, e_2 are untied edges of H, then they are also untied in G.*

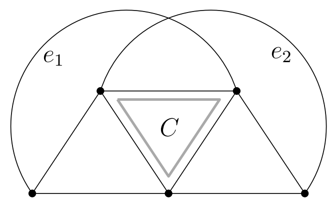
We introduce three families of signed graphs: hat, target, and hedgehog, each of which has a distinguished cycle C that is negative together with distinguished edges e_1, e_2. The edges e_1 and e_2 are untied in all. The heart of our argument is to show that if our graph is not one of the named counterexamples to Theorem 1.1, then it contains a hat, target, or hedgehog graph as a minor.



Hat



Target



Hedgehog

2.3 The main lemma and proof of main result

Our arguments lean on working with a carefully chosen negative cycle C in the graph. For this purpose we adopt Tutte's notation. Let G be a graph and let $H \subseteq G$. A *bridge* of H is a subgraph of $G - E(H)$ of one of the two forms: a single edge uv (and its ends) where $u, v \in V(H)$ and $uv \notin E(H)$, or a component F of $G - V(H)$ together with all edges of G with exactly one end in $V(F)$.

Lemma 2.3. *Let $G = (V, E, \sigma)$ be a simple, signed, 3-connected graph, and let $e_1, e_2 \in E(G)$ be nonadjacent. If there exists a negative cycle in $G - \{e_1, e_2\}$, then e_1 and e_2 are untied.*

Proof sketch. Suppose for contradiction the lemma is false, and let G be a counterexample so that $|V|$ is minimum. Choose a negative cycle $C \subseteq G - \{e_1, e_2\}$ subject to the following constraints: Both e_1 and e_2 are in the same bridge of C if possible, subject to this the bridge of C containing e_1 is maximum, subject to this the bridge of C containing e_2 is maximum, and subject to this the lexicographic ordering of the sizes of the other bridges is maximized. The proof proceeds by establishing the following four claims, whose proofs are omitted. They involve either a rerouting which contradicts the choice of C, or finding one of the minors in Section 2.2.

(1) Every bridge of C must contain e_1 or e_2.
(2) No bridge contains e_1 and e_2.
(3) e_1 is not incident with a vertex of C.
(4) $|V(C)| \geq 4$. □

With this lemma in hand, we prove the main result.

Proof sketch of Theorem 1.1. The "if" direction is straightforward to verify. For the "only if" direction, first suppose that e_1 and e_2 are incident with a common vertex u, say $e_i = uv_i$ for $i = 1, 2$. If $G - u$ is not balanced, then it contains a negative cycle that can be extended to a subgraph with a hat minor. Next, suppose that there exist two parallel edges f, f' of opposite sign. If f, f' are incident with an end of e_1 or e_2, then e_1, e_2 are not tied by 3-connectivity of G. Otherwise, the result follows from Theorem 1.2. This case also follows from an earlier result of Watkins and Mesner [11].

So we may now assume that G does not contain a negative cycle of length 2, and we may assume no parallel edges. If $G - \{e_1, e_2\}$ is balanced, then we have the third structure from the theorem statement. Otherwise, it follows from Lemma 2.3 that e_1 and e_2 are not tied in G, and this completes the proof. □

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