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Monochromatic configurations on a circle

(Extended abstract)

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Abstract

For $k \geq 3$, call a k-tuple $(d_1, d_2, \ldots, d_k)$ with $d_1 \geq d_2 \geq \cdots \geq d_k > 0$ and $\sum_{i=1}^{k} d_i = 1$ a *Ramsey k-tuple* if the following is true: in every two-colouring of the circle of unit perimeter, there is a monochromatic k-tuple of points in which the distances of cyclically consecutive points, measured along the arcs, are $d_1, d_2, \ldots, d_k$ in some order. By a conjecture of Stromquist, if $d_i = \frac{2^{k-i}}{2^k-1}$, then $(d_1, \ldots, d_k)$ is Ramsey.

Our main result is a proof of the converse of this conjecture. That is, we show that if $(d_1, \ldots, d_k)$ is Ramsey, then $d_i = \frac{2^{k-i}}{2^k-1}$. We do this by finding connections of the problem to certain questions from number theory about partitioning $\mathbb{N}$ into so-called *Beatty sequences*. We also disprove a majority version of Stromquist's conjecture, study a robust version, and discuss a discrete version.

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1 Introduction

In the May 2021 issue of the *American Mathematical Monthly*, Robert Tauraso posed the following problem [13]: *If all the points of the plane are arbitrarily coloured blue or red, find an acute pentagon with all vertices the same colour and with prescribed area* 1. A beautiful solution was suggested by Walter Stromquist, which reduced the question to a Ramsey-type problem, interesting on its own right.

Consider 31 points evenly spaced on a circle, and colour each of them arbitrarily blue or red. Then we can always find 5 points with the same colour that divide the circle into arcs proportional to 1:2:4:8:16. (The arcs need not be in the order suggested by the proportion. That is, 1:4:16:2:8 counts as a success.) Notice that no matter in what order 5 points divide the circle into such arcs, their convex hull is a pentagon of the same area. Thus, all we have to do is to start with a circle for which this area is 1. Stromquist managed to verify the above statement by computer, and he formulated the following attractive conjecture.

Conjecture 1.1 (Stromquist's conjecture)**.** *For any $k \geq 3$, consider $2^k - 1$ points evenly spaced on a circle, and colour each of them arbitrarily blue or red.*

Then we can always find k points with the same colour that divide the circle into arcs proportional to $1 : 2 : 4 : \ldots : 2^{k-1}$, but not necessarily in this order.

The case $k = 3$ was settled a long time ago by Bialostocki and Nielsen [4], and it is not hard to verify the case $k = 4$ either. Stromquist kindly informed us that he was able to give a computer assisted proof for $k \leq 6$.

In the present note, we study Stromquist's conjecture. To simplify the presentation, we introduce some notation. For $k \geq 3$, let $\underline{d} = (d_1, d_2, \ldots, d_k)$ be a k-tuple with $d_1 \geq d_2 \geq \cdots \geq d_k > 0$ and $\sum_{i=1}^{k} d_i = 1$. In a two-colouring of the circle S of unit perimeter, we call a k-tuple $(p_1, p_2, \ldots, p_k)$ of points from S *monochromatic* if the colour of every point p_i is the same. The main problem we study is whether for a given $\underline{d}$ it is true that in every two-colouring of S we can find a monochromatic k-tuple in which the distances of consecutive points, measured along the arcs, are exactly $d_1, \ldots, d_k$ in some order. We call a k-tuple $\underline{d}$ with this property a *Ramsey k-tuple*, or simply *Ramsey*.

A *permuted copy* of a k-gon inscribed in S is another k-gon inscribed in S with the same side lengths, but in a possibly different order. If the side lengths of the k-gon, measured along the arcs, are $d_1, \ldots, d_k$, we also call a monochromatic permuted copy of the k-gon a *monochromatic permuted copy*, or simply a *monochromatic copy*, of the k-tuple $\underline{d} = (d_1, d_2, \ldots, d_k)$.

Using this terminology, Stromquist's conjecture is equivalent to that if $k \geq 3$, and $d_i = \frac{2^{k-i}}{2^k - 1}$ for every $1 \leq i \leq k$, then $\underline{d} = (d_1, \ldots, d_k)$ is Ramsey. Our main result is proving the converse of the conjecture. That is, we prove that other k tuples are *not* Ramsey.

Theorem 1.2. *If $\underline{d} = (d_1, \ldots, d_k)$ is Ramsey, then $d_i = \frac{2^{k-i}}{2^k - 1}$.*

We call the k-tuple $\underline{d} = (d_1, \ldots, d_k)$ with $d_i = \frac{2^{k-i}}{2^k - 1}$ the $(k, 2)$-*power*. To prove Theorem 1.2, for every k-tuple $\underline{d}$ that is *not* the $(k, 2)$-power, we construct a two-colouring of

S that does not contain a monochromatic copy of $\underline{d}$. In fact, we show that for any other tuple $\underline{d}$ there exists a $t \in \mathbb{N}$, for which the colouring that consists of $2t$ arcs of equal length, coloured alternating red and blue, does not contain a monochromatic copy of $\underline{d}$. Theorem 1.2 is an immediate corollary of the following lemma, proved in Section 2.

Lemma 1.3. *Let c_t be a uniform colouring of S obtained by dividing it into $2t$ equal circular arcs, and colouring them alternating the two colours. If for every $t \in \mathbb{N}$ the uniform colouring c_t contains a monochromatic copy of $\underline{d} = (d_1, \dots, d_k)$, then $d_i = \frac{2^{k-i}}{2^k-1}$.*

Our proof proceeds by establishing a connection to a conjecture of Fraenkel about *Beatty sequences*, and solving a special case of it, which may be of independent interest.

A Beatty sequence is a sequence of the form $\{\lfloor \alpha n + \beta \rfloor\}_{n=1}^{\infty}$ for some $\alpha, \beta \in \mathbb{R}$. The term Beatty sequence was first used by Connell [5], after a problem proposed by Beatty [3]. Let $\underline{\alpha} = (\alpha_1, \dots, \alpha_k)$ with $0 < \alpha_1 \leq \dots \leq \alpha_k$ and $\underline{\beta} = (\beta_1, \dots, \beta_k)$ be two k-tuples of real numbers. We say that the pair $(\underline{\alpha}, \underline{\beta})$ *partitions* $\mathbb{N}$, if the Beatty sequences $\{\lfloor \alpha_i n + \beta_i \rfloor\}_{n=1}^{\infty}$ partition $\mathbb{N}$.

Finding a characterisation of those pairs $(\underline{\alpha}, \underline{\beta})$ which partition $\mathbb{N}$ is a well-studied problem, which has connections to a combinatorial game, called *Wythoff's game*, see for example [5, 6, 7, 8, 9, 15]. For $k = 2$, the characterisation is well understood [8, 11]. Fraenkel [8] noted that for $k \geq 3$ and for $\underline{\alpha} = (\alpha_1, \dots, \alpha_k)$ with $\alpha_i = \frac{2^k-1}{2^{k-i}}$ for every $1 \leq i \leq k$, there is a $\underline{\beta}$ such that $(\underline{\alpha}, \underline{\beta})$ partitions $\mathbb{N}$. According to Erdős and Graham,[1] Freankel made the following conjecture.

Conjecture 1.4 (Fraenkel's conjecture)**.** *If for $\underline{\alpha} = (\alpha_1, \dots, \alpha_k)$ with $k \geq 3$ and $0 < \alpha_1 < \dots < \alpha_k$ the pair $(\underline{\alpha}, \underline{\beta})$ partitions $\mathbb{N}$, then $\alpha_i = \frac{2^k-1}{2^{i-1}}$ for $1 \leq i \leq k$.*

Conjecture 1.4 is confirmed for $k \leq 7$ [1, 2, 10, 16, 14], and is open for $k \geq 8$. To prove Theorem 1.2, we prove Fraenkel's conjecture in a special case.

Theorem 1.5. *If $\alpha_i = \frac{\beta_i}{2}$ for every $1 \leq i \leq k$, and $(\underline{\alpha}, \underline{\beta})$ partitions $\mathbb{N}$, then $\alpha_i = \frac{2^k-1}{2^{k-i}}$ for every $1 \leq i \leq k$.*

We omit the details of the proof of Theorem 1.5 here, due to space restrictions.

In most of our proofs about Ramsey k-tuples, we work with a discrete version of the problem. We can do so because if there is an i for which $\frac{d_i}{\sum_j d_j}$ is irrational, then it is easy to show that $\underline{d}$ is not Ramsey. Indeed, we can to two-colour the points of S with no monochromatic pair of points at a given irrational distance apart.

Assuming $\sum_i d_i = 1$ and that every d_i is rational, then writing $d_i = \frac{p_i}{q_i}$ for every $1 \leq i \leq k$, for $N = \text{lcm}(q_1, \dots, q_k)$ the problem is equivalent to deciding if in any two-colouring of the vertices of a regular N-gon inscribed in S, we can find a monochromatic copy of $\underline{d}$. In other words, the problem is equivalent to deciding if in every two-colouring of $\mathbb{Z}_N$ we can find a monochromatic k-tuple in which the differences of cyclically consecutive elements

[1]This appears at [6, page 19] but the there cited paper [8] of Fraenkel only states a weaker conjecture, asserting that there are i, j with $i \neq j$ such that the ratio α_i/α_j is an integer.

are $N \cdot d_1, \dots, N \cdot d_k$ in some order. We find connections between certain transformations in the discrete version and avoiding monochromatic copies by using uniform colourings in the original version.

Considering Stromquist's conjecture, we could not answer the more specific question whether every uniform two-colouring of S contains a monochromatic copy of the $(k, 2)$-power, however, we confirmed this for very large values of k by a computer search. This more specific question is related to another problem from number theory, which has connections to vector balancing and combinatorial discrepancy; see Conjecture 5.1.

One might assume that if in a two-colouring one colour class is denser than the other, then it will contain a $(k, 2)$-power. However, this is false. Let $0 < \varepsilon < 1/80$, and divide S into 10 intervals of lengths $1/8-\varepsilon, 1/16+\varepsilon, 1/8-\varepsilon, 1/16+\varepsilon, 1/8-\varepsilon, 1/16+\varepsilon, 1/8-\varepsilon, 1/8+\varepsilon, 1/16-\varepsilon, 1/8+\varepsilon$ in this order, and colour them alternating red and blue, starting with red. Then the set of red points has total length $1/2 + 1/16 - 5\varepsilon > 1/2$, but a straight-forward case analysis shows that there is no red copy of a $(k, 2)$-power for $k \geq 8$.

We also study what happens when instead of a copy of $\underline{d}$, we only want to find a copy ε-close to it. Two k-tuples $(p_1, \dots, p_k)$ and $(p'_1, \dots, p'_k)$ in S are *ε-close* if $|p_1 - p'_1|, \dots, |p_k - p'_k| \leq \varepsilon$. A k-tuple of points $\underline{p} = (p_1, \dots, p_k)$ in S is an *ε-close copy of $\underline{d}$* if it is ε-close to a copy of $\underline{d}$. We call a k-tuple *nearly-Ramsey*, if for every $\varepsilon > 0$ in every two-colouring of S there is a monochromatic ε-close copy of $\underline{d}$.

We show the following.

Theorem 1.6. *If $d_1 = \frac{1}{2}$, or $\underline{d}$ is $(\frac{4}{7}, \frac{2}{7}, \frac{1}{7})$, $(\frac{5}{8}, \frac{1}{4}, \frac{1}{8})$, $(\frac{3}{4}, \frac{1}{6}, \frac{1}{12})$, $(\frac{7}{12}, \frac{1}{4}, \frac{1}{6})$, then (d_1, d_2, d_3) is nearly-Ramsey.*

We also conjecture that these are the only nearly-Ramsey triples.

2 Proof of Lemma 1.3

Proof. Assume that for every t the colouring c_t contains a monochromatic copy of $\underline{d}$. By symmetry, we may assume that this copy is red. Going around the points corresponding to this monochromatic copy in some cyclic order, we must jump over each blue interval. An arc of distance d_i with red endpoints jumps over $\lfloor td_i \rceil$ blue intervals, where $\lfloor x \rceil$ is the rounding of x to the closest integer. Thus, we must have $\sum_{i=1}^{k} \lfloor td_i \rceil = t$ for every $t \in \mathbb{N}$. This implies that for every $t > 0$ we have $\sum_{i=1}^{k} (\lfloor td_i \rceil - \lfloor (t-1)d_i \rceil) = t - (t-1) = 1$.

On the other hand, $\lfloor td_i \rceil - \lfloor (t-1)d_i \rceil$ is either 0 or 1 for each $1 \leq i \leq k$. For a fixed i, we have $\lfloor td_i \rceil - \lfloor (t-1)d_i \rceil = 1$ exactly when t is in the sequence $\{\lfloor (n + \frac{1}{2})\frac{1}{d_i} \rfloor\}_{n=1}^{\infty} = \{\lfloor n\frac{1}{d_i} + \frac{1}{2d_i} \rfloor\}_{n=1}^{\infty}$. Thus, the sequences $\{\lfloor n\frac{1}{d_i} + \frac{1}{2d_i} \rfloor\}_{n=1}^{\infty}$ must partition $\mathbb{N}$, and Theorem 1.5 implies that $d_i = \frac{2^{i-1}}{2^k - 1}$. □

3 Discrete version

Assume that every d_i is rational, $\sum_i d_i = 1$, and write $d_i = \frac{p_i}{q_i}$, and let $N = \mathrm{lcm}(q_1, \ldots, q_k)$. In $\mathbb{Z}_N$ a *copy* of $\underline{d} = (d_1, \ldots, d_k)$ is a k-tuple in which the distances of cyclically consecutive elements are $N \cdot d_1, \ldots, N \cdot d_k$ in some order. A colouring of $\mathbb{Z}_N$ is $\underline{d}$*-free* if it does not contain any monochromatic copy of $(d_1, \ldots, d_k)$.

Let $\chi : \mathbb{Z}_N \to \{\text{red}, \text{blue}\}$ be a colouring of $\mathbb{Z}_N$ and let $t \in \mathbb{Z}_N^*$ be such that $\gcd(t, N) = 1$. Let $\chi^t : \mathbb{Z}_N \to \{\text{red, blue}\}$ be defined by $\chi^t(x) = \chi(tx)$. It is a simple fact that χ is $(d_1, \ldots, d_k)$-free if and only if χ^t is $(t \cdot d_1, \ldots, t \cdot d_k)$-free.

It follows that finding a two-colouring of the vertices of a regular N-gon inscribed in S without a monochromatic copy of a triple $\underline{d} = (d_1, d_2, d_3)$ is equivalent to finding a colouring of it without a monochromatic copy of $\underline{d}^t = (d_1^t, d_2^t, d_3^t)$, where $d_i^t = \frac{t \cdot N \cdot d_i \bmod N}{N}$ if $(t \cdot N \cdot d_1 \bmod N) + (t \cdot N \cdot d_2 \bmod N) + (t \cdot N \cdot d_3 \bmod N) = N$, and $d_i^t = \frac{N - (t \cdot N \cdot d_i \bmod N)}{N}$ if $(t \cdot N \cdot d_1 \bmod N) + (t \cdot N \cdot d_2 \bmod N) + (t \cdot N \cdot d_3 \bmod N) = 2N$.

Notice that if $d_1^t, d_2^t, d_3^t \leq \frac{1}{2}$, then colouring a half-arc in S blue, and the other in red, avoids all monochromatic copies of $\underline{d}^t$. Thus, if there is a t such that $\gcd(t, N) = 1$ and $d_1^t, d_2^t, d_3^t \leq \frac{1}{2}$, then $\underline{d}$ is not Ramsey. The next claim explains how this property is closely related to avoiding copies by using uniform colourings. Here we omit its proof.

Claim 3.1. *If* $\gcd(t, N) = 1$*, then the uniform colouring* c_t *avoids all monochromatic copies of* $\underline{d}$ *if and only if* $d_1^t, d_2^t, d_3^t \leq \frac{1}{2}$.

4 Robust version

Theorem 1.6 states that additionally $(\frac{5}{8}, \frac{1}{4}, \frac{1}{8})$, $(\frac{3}{4}, \frac{1}{6}, \frac{1}{12})$, $(\frac{7}{12}, \frac{1}{4}, \frac{1}{6})$ and any triple with $d_1 = \frac{1}{2}$ are also nearly-Ramsey. We conjecture that there are no other nearly-Ramsey triples.

Conjecture 4.1. (d_1, d_2, d_3) *is nearly-Ramsey if and only if it is* $(\frac{4}{7}, \frac{2}{7}, \frac{1}{7})$, $(\frac{5}{8}, \frac{1}{4}, \frac{1}{8})$, $(\frac{3}{4}, \frac{1}{6}, \frac{1}{12})$, $(\frac{7}{12}, \frac{1}{4}, \frac{1}{6})$ *or a triple with* $d_1 = \frac{1}{2}$.

We sketch the proof of Theorem 1.6 and provide some supporting evidence for Conjecture 4.1. We recolour a point $p \in S$ with *black* if there is a red and a blue point in every neighbourhood of p. If a colouring of S is not monochromatic, then there is at least one black point. If we can find an ε-close copy of $\underline{d}$ such that it only has red and black points (or blue and black), then we can also find a 2ε-close copy of it with only red (or only blue) points, by slightly moving the black points of the corresponding triple in S.

Proof sketch of Theorem 1.6. The proof for $(\frac{5}{8}, \frac{1}{4}, \frac{1}{8})$, $(\frac{3}{4}, \frac{1}{6}, \frac{1}{12})$, $(\frac{7}{12}, \frac{1}{4}, \frac{1}{6})$ is by a case analysis of the possible colourings of a regular 8-gon/12-gon, respectively, with a black point.

For $d_1 = \frac{1}{2}$, we show that for every $\varepsilon > 0$ every red-blue colouring contains a monochromatic ε-close copy of a given triple (d_1, d_2, d_3) with $d_1 = \frac{1}{2}$. We may assume that the colouring is not monochromatic, otherwise the statement is trivial. Thus, we may assume the existence of a black point p. Let p' be the point diametrically opposite to p, and q and q' be two other diametrically opposite points, such that any three of the four points

p, p', q, q' form a copy of (d_1, d_2, d_3). By the pigeonhole principle, without loss of generality, we may assume that at most one of p', q, q' is blue. But then the other three points form a copy of (d_1, d_2, d_3) without a blue point. □

Let $\underline{d} = (d_1, d_2, d_3)$ be a triple that Conjecture 4.1 asserts to be not nearly-Ramsey. We believe that for any such $\underline{d}$, there is a uniform colouring c_t as in Lemma 1.3 that contains no monochromatic ε-close copies of $\underline{d}$. We call $t \in \mathbb{N}$ *suitable* if c_t contains no monochromatic copies of $\underline{d}$, and *nearly-suitable*, if c_t contains no monochromatic ε-close copies of $\underline{d}$. In c_t the black points are exactly the endpoints of the intervals. Thus, t is nearly-suitable if and only if it is suitable and c_t avoids copies of $\underline{d}$ with two points coinciding with endpoints of the segments. As the distance of any two black points (along the circumference) is a multiple of $\frac{1}{2t}$, we obtain the following observation.

Observation 4.2. *A suitable t is nearly-suitable if and only if none of $2td_1, 2td_2, 2td_3$ is an integer.*

If one of d_1, d_2, d_3 is irrational, then any suitable t is also nearly-suitable, thus such (d_1, d_2, d_3) is not nearly-Ramsey. Otherwise, we write $d_i = \frac{p_i}{q_i}$ such that p_i, q_i are integers and $\gcd(p_i, q_i) = 1$ for $i = 1, 2, 3$. To prove Conjecture 4.1, it is thus sufficient to find a suitable t in $T := \{t : q_1, q_2, q_3 \nmid 2t\}$. We can prove that there is such t if q_1, q_2, q_3 are all odd, as well as in several other cases, but here we omit these proofs.

5 $(k, 2)$-powers in uniform colourings

We conjecture that for every t, the uniform colouring of c_t from Lemma 1.3 contains a monochromatic copy of the $(k, 2)$-power for every k. We have seen in the proof of Lemma 1.3 that in this case the sides of a red copy 'jump' over the t blue intervals. However, this is only a necessary condition, and not a sufficient one. Indeed, if a jump starts from a 'bad' part of a red interval, it might end up inside a blue one. More precisely, we can consider the problem as follows.

Let $N = 2t(2^k - 1)$ and colour vertices of a regular N-gon such that t reds are followed by t blues in an alternating manner, so that vertices $0, \ldots, 2^k - 1$ are red, $2^k, \ldots, 2 \cdot 2^k - 1$ are blue, $2 \cdot 2^k, \ldots, 3 \cdot 2^k - 1$ are red etc. If there is a monochromatic copy of the $(k, 2)$-power, there is also a red copy. For each vertex of the red copy of the $(k, 2)$-power, consider its index modulo $2^{k+1} - 2$. Each of these needs to be at most $2^k - 1$. Moreover, the differences among the consecutive vertices need to be $2t \pmod{2^{k+1} - 2}$, $4t \pmod{2^{k+1} - 2}$, $\ldots$, $2^k t \pmod{2^{k+1} - 2}$, in some order. To have such a k-tuple of indices modulo $2^{k+1} - 2$ is a necessary and sufficient condition for the existence of a red copy.

By computer, we verified this up to a large k. We phrase a problem in a more natural and general form. Interpret the numbers $2^i t \pmod{2^{k+1} - 2}$ that are larger than $2^k - 1$ as $2^{k+1} - 2 - 2^i t$, and denote these k numbers by $v_1, \ldots, v_k$. With this, the numbers v_i will determine how one vertex moves compared to the preceding vertex in the $0, \ldots, 2^k - 1$ interval. Note that none of these numbers can be equal to $2^k - 1$. Thus, $-2^k + 1 < v_1, \ldots, v_k < 2^k - 1$, and $\sum_{i=1}^{k} v_k = 0$, since the k-gon with these side-distances exists.

We get the following even nicer question if we divide by $2^k - 1$.

Conjecture 5.1. *If a sequence of reals $-1 < x_1, \ldots, x_k < 1$ satisfies*

$$x_{i+1} = \begin{cases} 2x_i, & \text{if } 2|x_i| < 1 \\ 2x_i - 2, & \text{if } 2x_i > 1 \\ 2 - 2x_i, & \text{if } 2x_i < -1 \end{cases}$$

for $i = 1, \ldots, k$, such that $x_{k+1} = x_1$, then there is a permutation π of $\{1, \ldots, k\}$ such that $0 \le \sum_{i=1}^{j} x_\pi(i) < 1$ for every j.

This conjecture is similar to Steinitz's theorem [12], and to other vector balancing problems. Indeed, it can be proved for any x_i's satisfying the conditions of the conjecture, $\sum_{i=1}^{k} x_i = 0$. We note that if the x_i's are any sequence satisfying $\sum_{i=1}^{k} x_i = 0$ and $|x_i| < 1/2$ for every i, then one can easily find a permutation for which $0 \le \sum_{i=1}^{j} x_\pi(i) < 1$ for every j. But without this bound, we have to exploit that $x_{i+1} = 2x_i$, as otherwise there would be counterexamples, i.e., $0.6, 0.6, 0.6, -0.9, -0.9$. Could it be that the conjecture is true because we always have many i's such that $|x_i| < 1/2$, and these can be used somehow to take care of the other x_i's?

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