

PROVING A DIRECTED ANALOGUE OF THE GYÁRFÁS-SUMNER CONJECTURE FOR ORIENTATIONS OF P_4

(EXTENDED ABSTRACT)

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Abstract

An oriented graph is a digraph that does not contain a directed cycle of length two. An (oriented) graph D is H -free if D does not contain H as an induced sub(di)graph. The Gyárfás-Sumner conjecture is a widely-open conjecture on simple graphs, which states that for any forest F , there is some function f such that every F -free graph G with clique number $\omega(G)$ has chromatic number at most $f(\omega(G))$. Aboulker, Charbit, and Naserasr [Extension of Gyárfás-Sumner Conjecture to Digraphs; E-JC 2021] proposed an analogue of this conjecture to the dichromatic number of oriented graphs. The *dichromatic number* of a digraph D is the minimum number of colors required to color the vertex set of D so that no directed cycle in D is monochromatic.

Aboulker, Charbit, and Naserasr's $\vec{\chi}$ -boundedness conjecture states that for every oriented forest F , there is some function f such that every F -free oriented graph D has dichromatic number at most $f(\omega(D))$, where $\omega(D)$ is the size of a maximum

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clique in the graph underlying D . In this paper, we perform the first step towards proving Aboulker, Charbit, and Naserasr's $\overrightarrow{\chi}$ -boundedness conjecture by showing that it holds when F is any orientation of a path on four vertices.

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1 Introduction

In a simple graph, the size of a maximum clique gives a lower bound on its chromatic number. But if a graph contains no large cliques, does it necessarily have small chromatic number? This question has been answered in the negative. In 1959, Erdős showed that there exist graphs with arbitrarily high girth and arbitrarily high chromatic number [10]. Hence, if a graph H contains a cycle there are graphs with arbitrarily high chromatic number and no induced copy of H . Around the 1980s, Gyárfás and Sumner independently conjectured [13, 25] that for any forest H , all graphs with bounded clique number and no induced copy of H have bounded chromatic number. The conjecture has been proven for some specific classes of forests but remains largely open; see [23] for a survey of related results. This paper concerns an extension of the Gyárfás-Sumner conjecture to directed graphs proposed by Aboulker, Charbit, and Naserasr [3].

We call a digraph *oriented* if it has no *digon* (directed cycle of length two). This paper will focus on finite, simple, oriented graphs. For a digraph $D = (V, E)$ we define the *underlying graph* of D to be the graph $D^* = (V, E^*)$ where E^* is the set obtained from E by replacing each arc $e \in E$ by an undirected edge between the same two vertices. We say two vertices in D are *adjacent* or *neighbors* if they are adjacent in D^* . We denote the set of neighbors of a vertex $v \in V(D)$ by $N(v)$ and we denote $N(v) \cup \{v\}$ by $N[v]$. For a set of vertices $S \subseteq V(D)$ we let $N(S)$ and $N[S]$ denote the sets $\cup_{v \in S} N(v) \setminus S$ and $\cup_{v \in S} N[v]$. For a subdigraph $H \subseteq D$ we let $N(H)$ denote the set $N(V(H))$. We let P_t denote the path on t vertices and $\overrightarrow{P}_t$ be the path $p_1 \rightarrow p_2 \rightarrow \cdots \rightarrow p_t$. We call an oriented graph whose underlying graph is a clique a *tournament*. Given a (di)graph G and $S \subseteq V$, we denote the sub(di)graph of G induced by S as $G[S]$. We say that a (di)graph G *contains* a (di)graph H if G contains H as an induced sub(di)graph. If G does not contain a (di)graph H we say that G is *H -free*. The *clique number* and the *chromatic number* of a digraph are the chromatic number and clique number of its underlying graph, respectively. We denote the clique number and the chromatic number of a (di)graph G by $\omega(G)$ and $\chi(G)$, respectively. We say that a graph H is *χ -bounding* if there exists a function f with the property that every H -free graph G satisfies $\chi(G) \leq f(\omega(G))$. In this language, the Gyárfás-Sumner conjecture states that every forest is χ -bounding.

How can the Gyárfás-Sumner conjecture be adapted to the directed setting? A first idea is to call an oriented graph H *χ -bounding* if there exists a function f with the property that every H -free oriented graph D satisfies $\chi(D) \leq f(\omega(D))$. Then, once again, by [10], all χ -bounding oriented graphs are oriented forests. Note that if an oriented graph H is χ -bounding, its underlying graph H^* is also χ -bounding. However, the converse does not

hold, as, for instance, P_4 is χ -bounding, but there exist orientations of P_4 that are not χ -bounding. There are four different orientations of P_4 , up to reversing the order of the vertices on the whole path:

$$\vec{P}_4 : \rightarrow \rightarrow \rightarrow, \vec{A}_4 : \rightarrow \leftarrow \rightarrow, \vec{Q}_4 : \rightarrow \leftarrow \leftarrow, \vec{Q}_4' : \leftarrow \leftarrow \rightarrow$$

Chudnovsky, Scott and Seymour showed $\vec{Q}_4$ and $\vec{Q}_4'$ are χ -bounding in [8]. However, $\vec{P}_4$ and $\vec{A}_4$ are not $\vec{\chi}$ -bounding as shown by Kierstead and Trotter [18] and Gyárfás [15], respectively. Chudnovsky, Scott and Seymour [8] showed that $\vec{Q}_4, \vec{Q}_4'$ are both χ -bounding in 2019. In the same article, the authors show that orientations of stars are also χ -bounding.

Our first attempt at adapting the Gyárfás-Sumner conjecture to oriented graphs failed for oriented paths such as $\vec{P}_4$ and $\vec{A}_4$. Hence, we focus on a different approach proposed by Aboulker, Charbit, and Naserasr [3] which uses a concept called “dichromatic number” introduced in [21]. A *dicoloring* of a digraph D is a partition of $V(D)$ into classes, or colors, such that each class induces an acyclic digraph (that is, there is no monochromatic directed cycle). The *dichromatic number* of D , denoted as $\vec{\chi}(D)$, is the minimum number of colors needed for a dicoloring of D . Notice that every coloring of a directed graph D is also a dicoloring, thus $\vec{\chi}(D) \leq \chi(D)$. We say a class of digraphs $\mathcal{D}$ is $\vec{\chi}$ -*bounded* if there exists a function f such that every $D \in \mathcal{D}$ satisfies $\vec{\chi}(D) \leq f(\omega(D))$ and we call such an f a $\vec{\chi}$ -*binding function* for $\mathcal{D}$. We say that a digraph H is $\vec{\chi}$ -*bounding* if the class of H -free oriented graphs is $\vec{\chi}$ -bounded.

We can now state Aboulker, Charbit, and Naserasr’s dichromatic analogue to the Gyárfás-Sumner conjecture for digraphs. For brevity, we will call this conjecture the “ACN $\vec{\chi}$ -boundedness” conjecture in the remainder of this extended abstract.

Conjecture 1.1 (The ACN $\vec{\chi}$ -boundedness conjecture [3]). *Every oriented forest is $\vec{\chi}$ -bounding.*

The converse of the ACN $\vec{\chi}$ -boundedness conjecture holds; all $\vec{\chi}$ -bounding digraphs must be oriented forests. Indeed, Harutyunyan and Mohar proved that there exist oriented graphs of arbitrarily large undirected girth and dichromatic number [16]. Oriented graphs of sufficiently large undirected girth (and no digon) forbid any fixed digraph that is not an oriented forest.

The ACN $\vec{\chi}$ -boundedness conjecture is still widely open. It is not known whether the conjecture holds for any orientation of any tree T on at least five vertices that is not a star. In particular, it is not known whether the conjecture holds for oriented paths. In contrast, Gyárfás showed that every path is χ -bounding in the 1980’s [13, 14] via short and elegant proof. For $t \leq 3$, every orientation of P_t is trivially $\vec{\chi}$ -bounding. However, for $t \geq 4$, the picture gets more complicated. Let T be any fixed orientation of K_3 . In [3], Aboulker, Charbit and Naserasr showed that class of $(T, \vec{P}_4)$ -free oriented graphs have bounded dichromatic number. The authors also show that $\vec{P}_4$ -free oriented graphs with clique number at most three have bounded dichromatic number. Recently, Aboulker,

Aubian, Charbit, and Thomassé showed that $\vec{P}_6$ -free oriented graphs with clique number at most two also have bounded dichromatic number [1]. See [5] for further related results.

Let $\vec{K}_t$ denote the transitive tournament on t vertices. Steiner showed that the class of $(\vec{K}_3, \vec{A}_4)$ -free oriented graphs has bounded dichromatic number in [24]. In the same paper Steiner asked whether the class of $(H, \vec{K}_t)$ -free oriented graphs has bounded dichromatic number for $t \geq 4$ and $H \in \{\vec{P}_4, \vec{A}_4\}$. Our main result answers this question in the affirmative as corollary.

1.1 Our contributions

In this paper, we show that every orientation of P_4 is $\vec{\chi}$ -bounding and thus the ACN $\vec{\chi}$ -boundedness conjecture holds for all orientations of P_4 . The ACN $\vec{\chi}$ -boundedness conjecture is open for any orientation of P_t for $t \geq 5$. Our main novel result is that $\vec{P}_4$ and $\vec{A}_4$ are both $\vec{\chi}$ -bounding. To summarize, our main result is the following:

Theorem 1.2. *Let H be an oriented P_4 . Then, the class of H -free oriented graphs is $\vec{\chi}$ -bounded. In particular, for any H -free oriented graph D ,*

$$\vec{\chi}(D) \leq (\omega(D) + 7)^{(\omega(D) + 8.5)}.$$

2 Proof Sketch

In this section we will sketch the proof of Theorem 1.2. The full proof is available in the arXiv version of this paper [9]. Our main tool in the proof is an object called a “dipolar set” which was first introduced in [2] as a “nice set”.

Definition 2.1. *A dipolar set of an oriented graph D is a nonempty subset $S \subseteq V(D)$ that can be partitioned into S^+, S^- such that no vertex in S^+ has an out-neighbor in $V(D \setminus S)$ and no vertex in S^- has an in-neighbor in $V(D \setminus S)$.*

We will use the following lemma from [2] which reduces the problem of bounding the dichromatic number of D to bounding the dichromatic number of a dipolar set in every induced oriented subgraph of D .

Lemma 2.2 (Lemma 17 in [2]). *Let $\mathcal{D}$ be a family of oriented graphs closed under taking induced subgraphs. Suppose there exists a constant c such that every $D \in \mathcal{D}$ has a dipolar set S with $\vec{\chi}(S) \leq c$. Then every $D \in \mathcal{D}$ satisfies $\vec{\chi}(D) \leq 2c$.*

We will give a way of finding a dipolar set in any oriented graph excluding some orientation of P_4 as an induced subdigraph and show how to bound its dichromatic number. The backbone of our dipolar set is an object we call a closed tournament.

Definition 2.3. *We say K and P form a closed tournament $C = K \cup V(P)$ if K is a tournament of maximum order and P is a directed path from a source component to a sink*

component of the directed graph induced by K . We say K and P form path-minimizing closed tournament if $|P|$ is minimized amongst all choices of K, P that form a closed tournament.

Lemma 2.4. *Let H be an orientation of P_4 and D be an H -free oriented graph. Let C be a closed tournament in D and let X be the set of vertices with both an in-neighbor and an out-neighbor in C . Then $N[C \cup X]$ is a dipolar set.*

The proof follows from the fact that every $v \in N(C)$ must have a non-neighbor in C and from the definition of strong connectivity. See Lemma 3.1 in [9] for details.

Our proof that orientations of P_4 are $\vec{\chi}$ -bounding proceeds by induction. Let H be an oriented P_4 and let $\omega > 1$ be an integer. We let γ be the maximum of $\vec{\chi}(D')$ over every H -free oriented graph D' satisfying $\omega(D') < \omega$. We assume γ is finite. We let D be an H -free oriented graph with clique number ω and assume D is strongly connected.

Observation 2.5. *Every $v \in V(D)$ satisfies $\vec{\chi}(N(v)) \leq \gamma$.*

Let $C = K \cup V(P)$ be a path-minimizing closed tournament in D . Let X be the set of vertices in $N(C)$ with an in-neighbor and an out-neighbor in C . Then $N[C \cup X]$ is a closed tournament. It remains to show that $\vec{\chi}(N[C \cup X])$ is bounded by a function of ω and γ .

By Observation 2.5, $\vec{\chi}(N[K]) \leq \omega \cdot \gamma + \omega$. Let the vertices of P be $p_1 \rightarrow p_2 \rightarrow \cdots \rightarrow p_\ell$, in order. Then since C is path minimizing, P is a shortest directed path from p_1 to p_ℓ . Hence:

Observation 2.6. *For each integer $2 \leq i + 1 < j \leq \ell$, there is no arc from p_i to p_j .*

It follows that $\vec{\chi}(V(P)) \leq 2$. Hence, it is enough to show that $\vec{\chi}(N(P) \setminus N[K])$ and $\vec{\chi}(N(X) \setminus N[C])$ are bounded by a function of ω and γ . We obtain that $\vec{\chi}(N(X) \setminus N[C]) \leq 2\gamma$ by applying the following lemma with $Q := C$, $R := X$ and $S := N(X) \setminus N[C]$.

Lemma 2.7. *Let H be an oriented P_4 and let D be an H -free oriented graph. Suppose there is a partition of $V(D)$ into sets Q, R, S such that there is no arc between Q and S , every $r \in R$ has both an in-neighbor and an out-neighbor in Q and every $s \in S$ has a neighbor in R . Let γ be an integer such that for every $r \in R$, we have $\vec{\chi}(N(r)) \leq \gamma$. Then $\vec{\chi}(S) \leq 2\gamma$.*

Lemma 2.7 follows from an easy inductive argument on $|S|$. (See Lemma 4.3 in [9] for details). By Lemma 2.7, it only remains to show that $\vec{\chi}(N(P) \setminus N[K])$ is bounded by some function of γ and ω . Note, we cannot simply apply Observation 2.5 because P may be arbitrarily long. For this part of the proof we proceed (slightly) differently for each orientation of P_4 . Here, we present a sketch of the case when $H = \vec{P}_4$. The other cases are similar.

We say the “first” and “last” neighbors of a vertex $v \in N(P)$ are the vertices $p_i \in N(v) \cap V(P)$ minimizing i and maximizing i , respectively. For each integer $1 \leq i \leq \ell$, we let F_i, L_i denote the sets of vertices in $N(P)$ whose first neighbor is p_i and whose last neighbor is p_i , respectively. Let F_i^-, L_i^+ be the sets consisting of all out-neighbors of p_i in F_i and in-neighbors of p_i in L_i , respectively.

Observation 2.8. *Let $2 \leq i < j \leq \ell - 1$. Then there are no arcs from F_i^- to F_j and no arcs from L_i to L_j^+ .*

Indeed, as otherwise $D[N[\{p_i, p_j\}]]$ would contain a $\vec{P}_4$. Let $W = \cup_{i=2}^{\ell-1} (F_i^- \cup L_i^+)$. Then by Observation 2.5 and Observation 2.8, $\vec{\chi}(W) \leq 2\gamma$. Let $R = N(P) \setminus (W \cup N(\{p_1, p_2, p_\ell\}))$. By Observation 2.5, we need only show showing $\vec{\chi}(R)$ is at most some function of γ and ω to complete the proof of Theorem 1.2. We will require a technical lemma:

Lemma 2.9. *Let $v, w \in R$, if $(w, v) \in E(D)$, there is a directed path from v to w on at most $\max\{6, \ell - 1\}$ vertices.*

Lemma 2.9 follows from Observation 2.6 and a brief case analysis. See Lemma 5.4 from [9] for details.

Lemma 2.10. $\vec{\chi}(R) \leq 6\gamma$.

Proof. If P contains at most six vertices then $\vec{\chi}(N(P)) \leq 6\gamma$, hence we may assume this is not the case. We may assume that there is a tournament J of size ω in $D[R]$ for otherwise $\vec{\chi}(R) \leq \gamma$. Since $P \neq \emptyset$ and $C = K \cup V(P)$ was chosen to be path-minimizing it follows that J cannot be strongly connected. Let v be a vertex in the sink component of J and w be a vertex in the source component of J . Therefore, $(w, v) \in E(D)$. Thus by Lemma 2.9 there is a path Q from v to w of length less than that of P . Hence, J, P' form a closed tournament. By definition since K, P were chosen to form a path-minimizing closed tournament P' cannot be shorter than P , a contradiction. $\square$

By combining the results from this section, we obtain that $\vec{\chi}(N[C \cup X])$ is at most some function of γ and ω . Since $N[C \cup X]$ is dipolar, it follows from Lemma 2.2 that $\vec{\chi}(D)$ is at most some function of γ and ω . Hence, by induction $\vec{P}_4$ is $\vec{\chi}$ -binding. The proofs that $\vec{A}_4, \vec{Q}_4$ and $\vec{Q}_4'$ are $\vec{\chi}$ -binding are similar (and slightly simpler). Full details can be found in [9].

3 Conclusion

Our result is an initial step towards resolving the ACN $\vec{\chi}$ -boundedness conjecture for orientation of paths in general. However, we think we are still far from this result. It would already be interesting to hear the answer to the easier question: Is it true that for every oriented path H there is a constant c_H such that every oriented graph not containing H or a tournament of size three has dichromatic number at most c_H . By Theorem 1.2 this is known when H is an orientation of a path of length at most four. It is proven in [1] that this is true when $H = \vec{P}_6$.

Recall that the classes of $\vec{Q}_4$ -free oriented graphs and $\vec{Q}_4'$ -free oriented graphs were already shown to be χ -bounded in [8]. The χ -binding function f' for these two classes from [8] is defined using recurrence $f'(x) := 2(3f'(x-1))^5$ which leads to a double-exponential

bound on χ , and cannot guarantee a better bound on $\vec{\chi}$. In this paper, Theorem 1.2 provides an improved $\vec{\chi}$ -binding function for these classes. We would like to know whether any orientation of P_4 is *polynomially* $\vec{\chi}$ -bounding. In other words, is there some oriented P_4 so that the class of oriented graphs forbidding it has a *polynomial* $\vec{\chi}$ -binding function?

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