

A DIRECT BIJECTION BETWEEN TWO-STACK SORTABLE PERMUTATIONS AND FIGHTING FISH

(EXTENDED ABSTRACT)

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Abstract

We define a bijection between *two-stack sortable permutations* and *fighting fish*, enriching the garden of bijections linking the numerous combinatorial classes counted by the sequence A000139 of the OEIS. Our bijection is (up to symmetry) the non-recursive version of the one of Fang (2018). Along the way, we encounter *labeled sorting trees*, a new class of trees that appear to have nice properties that seem worth to explore.

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1 Introduction

The problem of sorting a permutation through a stack was addressed by Knuth in his seminal work [9] (Section 2.2.1), initiating the development of the field of pattern-avoiding permutations. While one-stack sortable permutations are characterized by their avoidance of the pattern 231 and counted by the Catalan numbers, West [11] established the pattern-avoidance characterization of two-stack sortable permutations, but did not succeed in proving that the number of such permutations of size n is given by the nice formula $\frac{2}{(n+1)(2n+1)}\binom{3n}{n}$, leaving it as a conjecture. This formula was first proved by Zeilberger [12] using generating functions, then refined according to certain statistics by Bousquet-Mélou [1]. Other proofs of this enumeration were later found by Dulucq, Gire and Guibert [6], by Goulden and West [8] establishing bijections between two-stack sortable permutations and nonseparable planar maps using either generating trees or recursive decompositions. Hence

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two combinatorial explanations of the enumeration formula were found, but they were still not completely satisfying because recursive. More recently, Fang [7] used another recursive decomposition of two-stack sortable permutations to define a bijection with fighting fish, a generalization of parallelogram polyominoes introduced in 2016 by Duchi, Guerrini, Rinaldi and Schaeffer [2]. The main contribution of the present paper is a direct description of the bijection of Fang using a particular class of trees that we call labeled sorting trees, which, combined with the encoding of nonseparable planar maps by fighting fish given by Duchi and the third author [4], leads to a direct bijective path from two-stack sortable permutations to nonseparable planar maps. This is one step further in the bijective understanding of the connections between combinatorial structures enumerated by [10, A000139]: we refer to the first page of [4] for a diagram summarizing known bijections.

The paper is organized as follows: we first present the objects involved in our bijection (two-stack sortable permutations and fighting fish), we then describe our bijection going through labeled sorting trees, and we finish by an overview of open questions arising naturally in our work.

2 Preliminaries

2.1 Stack sorting on permutations

Let τ be a word made of distinct positive integers. If τ is empty, we define $\mathbf{S}(\tau) = \tau$. Else, if we denote by k the largest letter of τ , we can write $\tau = \tau_1 k \tau_2$, and we then define $\mathbf{S}(\tau) = \mathbf{S}(\tau_1) \mathbf{S}(\tau_2) k$. A permutation $\sigma \in \mathfrak{S}_n$ is identified with its one-line notation $\sigma_1 \dots \sigma_n$, where we write σ_i for $\sigma(i)$. Restricted to permutations, \mathbf{S} is said to be the operator of *stack-sorting*. Stack-sorting deserves its name because $\mathbf{S}(\sigma)$ is the word τ obtained from σ via the following procedure on a stack constrained to be decreasing from bottom to top. Initialize the stack to be empty and τ to be the empty word, and at each step, consider the smallest i such that σ_i has not yet been put in the stack : if the stack is empty or if the top element of the stack is greater than σ_i then put σ_i on the top of the stack, otherwise pop the top element of the stack and append it to τ . If all elements of σ have been treated, then pop out the top of the stack and append it to τ . The procedure ends when all elements of σ have been treated and the stack is empty. We can encode this procedure by a Dyck path : we add an up step $u = (1, 1)$ to the path each time we put an element on the top of the stack, and a down step $d = (1, -1)$ each time we pop out the top element of the stack. We denote by $D(\sigma)$ the Dyck path obtained for a permutation $\sigma \in \mathfrak{S}_n$: it is a path from $(0, 0)$ to $(2n, 0)$ staying weakly above the x -axis. For $k \geq 1$, we define $D_k(\sigma) = D(\mathbf{S}^{k-1}(\sigma))$.

For $k \geq 0$, a permutation $\sigma \in \mathfrak{S}_n$ is said to be *k-stack sortable* if $\mathbf{S}^k(\sigma)$ is the identity. A *k-stack sortable* permutation σ is uniquely determined by the tuple $(D_1(\sigma), \dots, D_k(\sigma))$ because σ can be recovered from the identity by reverting the stack-sorting process encoded by this tuple of Dyck paths. We denote by $2\mathcal{SS}_n$ the set of two-stack sortable permutations

of \mathfrak{S}_n , and $2\mathcal{SS} = \bigcup_{n \geq 1} 2\mathcal{SS}_n$.

2.2 Fighting fish

While fighting fish have been introduced in [2] in terms of gluings of square cells, we present them here as words on the alphabet $\{E, N, W, S\}$ (see Figure 1):

Definition 1. A word $w \in \{E, N, W, S\}^*$ is a fighting fish if it can be obtained from the word $ENWS$ using a finite sequence of the following 3 operations:

- Upper gluing : replace a subword W by NWS .
- Lower gluing : replace a subword N by ENW .
- Double gluing : replace a subword WN by NW .

The size of a fighting fish is half of its length minus 1. We denote by \mathcal{FF}_n the set of fighting fish of size n , and by $\mathcal{FF} = \bigcup_{n \geq 1} \mathcal{FF}_n$.

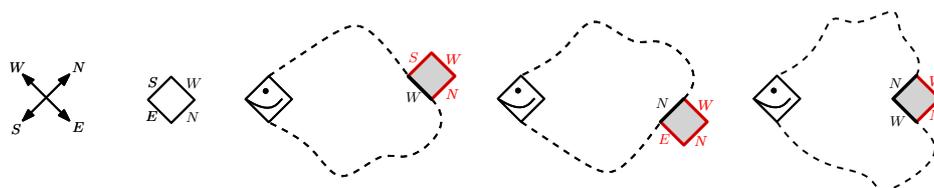


Figure 1: The tilted cardinal steps, a cell and operations of upper, lower and double gluing.

There is a natural notion of symmetry on fighting fish: if F is a fighting fish, its *conjugate*, denoted $\text{Conj}^{\mathcal{FF}}(F)$, is the fighting fish obtained by reversing F and changing its letters with the rules $E \leftrightarrow S, W \leftrightarrow N$. Conjugation is an involution on fighting fish, that is $\text{Conj}^{\mathcal{FF}} \circ \text{Conj}^{\mathcal{FF}}$ is the identity on \mathcal{FF} , and we can see it as the symmetry with respect to the horizontal axis on our two-dimensional pictures.

3 The bijection from two-stack sortable permutations to fighting fish

3.1 From permutations to labeled sorting trees

Let $\sigma \in \mathfrak{S}_n$ be a permutation, and let us define $\hat{\sigma} \in \mathfrak{S}_{n+1}$ by setting $\hat{\sigma}(1) = n + 1$ and $\hat{\sigma}(i) = \sigma(i - 1)$ for $2 \leq i \leq n + 1$. We represent $\hat{\sigma}$ as the set of points $\{(i, \hat{\sigma}(i))\}$ in \mathbb{Z}^2 (its *grid representation*) and we construct a rooted plane tree on this set of points. The *sorting tree* $\text{ST}(\sigma)$ associated to σ is the rooted plane tree obtained by the following top-to-bottom process :

- Define the root to be $(1, n + 1)$.
- At step $j \geq 1$, we insert the point $(k, \hat{\sigma}(k))$ in the tree, where k is such that $\hat{\sigma}(k) = n - j$. To do so, let us consider $0 = i_1 < i_2 < \dots < i_j$ the x -coordinates of all points already inserted in the tree. There is then a maximal index m such that $i_m < k$. We distinguish two cases :
 - If $m = j$ or $\hat{\sigma}(i_m) < \hat{\sigma}(i_{m+1})$, we define the parent of $(k, \hat{\sigma}(k))$ to be $(i_m, \hat{\sigma}(i_m))$.
 - If $m < j$ and $\hat{\sigma}(i_m) > \hat{\sigma}(i_{m+1})$, we consider the greatest $m + 1 \leq r \leq n$ such that $\hat{\sigma}(i_m) > \hat{\sigma}(i_{m+1}) > \dots > \hat{\sigma}(i_r)$, and we set the parent of $(k, \hat{\sigma}(k))$ to be $(i_r, \hat{\sigma}(i_r))$.
- The process ends when all points have been inserted, i.e. after the n^{th} step.

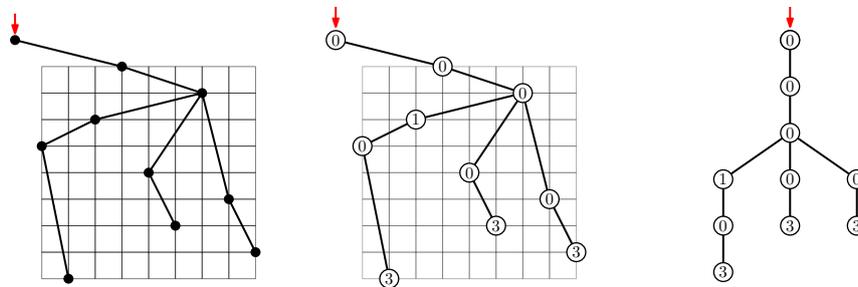


Figure 2: The permutation 617953842, its sorting tree and its labeled sorting tree

This procedure produces a tree since n edges are inserted and no cycles are created, because each non-root vertex has a parent of strictly greater y -coordinate. The permutation $\hat{\sigma}$ can be split into maximal descending runs in a unique way. We associate to every element of $\hat{\sigma}$ its *run label* in the following way: if it is not the last element of its descending run, we label it by 0, else we label it by the number of elements in its descending run. The *labeled sorting tree* $LST(\sigma)$ associated to σ is the plane rooted tree obtained by labeling each node of $ST(\sigma)$ with the run label of the element to which it corresponds in the permutation. As an example, we give the (labeled) sorting tree of the permutation 617953842 in Figure 2.

Sorting trees and labeled sorting trees of permutations are intimately linked to the Dyck paths corresponding to their first two passes into a stack:

Proposition 1. *Let $\sigma, \tau \in \mathfrak{S}_n$. Then:*

- $ST(\sigma) = ST(\tau)$ if and only if $D_2(\sigma) = D_2(\tau)$.
- $LST(\sigma) = LST(\tau)$ if and only if $(D_1(\sigma), D_2(\sigma)) = (D_1(\tau), D_2(\tau))$.

In particular, when $\sigma, \tau \in 2SS_n$, $ST(\sigma) = ST(\tau)$ if and only if $\mathbf{S}(\sigma) = \mathbf{S}(\tau)$.

The proof of the next proposition, that we do not present here, relies mainly on the decomposition of two-stack sortable permutations presented in [7] and on its isomorphic counterpart on labeled sorting trees, transferred by LST:

Proposition 2. *Let T be a rooted labeled plane tree with root r , and having n non-root vertices (we say that it has size n). For a given node $v \in T$, we denote by $\lambda(v)$ its nonnegative label, $\deg(v)$ its number of children, $\text{sub}(v)$ the subtree of T rooted at v and $\text{anc}(v)$ the nodes w such that v belongs to $\text{sub}(w)$ (the ancestors of v). Then there exists a permutation $\sigma \in \mathfrak{S}_n$ such that $\text{LST}(\sigma) = T$ if and only if:*

$$\begin{cases} \sum_{v \in T} \lambda(v) = n + 1 \\ \forall v \in T, \lambda(v) \leq \sum_{w \in \text{anc}(v)} (2 - \deg(w)) - 1 \\ \forall v \in T \setminus \{r\}, \sum_{w \in \text{sub}(v)} (\lambda(w) - 1) \geq 1 \end{cases}$$

Furthermore, denoting by \mathcal{LST}_n the set of trees satisfying these three conditions, the restriction of the map LST to two-stack sortable permutations is a bijection between $2SS_n$ and \mathcal{LST}_n .

Since $\text{LST}(\mathfrak{S}_n) = \mathcal{LST}_n$, we can call trees in $\mathcal{LST} = \bigcup_{n \geq 1} \mathcal{LST}_n$ labeled sorting trees.

3.2 From labeled sorting trees to fighting fish

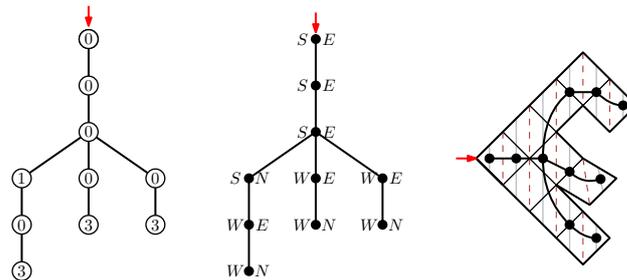


Figure 3: A tree in \mathcal{LST}_9 , its fish word displayed on the tree and the corresponding fighting fish.

Let T be a tree in \mathcal{LST}_n . We build a word w on the alphabet $\{E, N, W, S\}$ with the following algorithm using a stack :

- Set w to be the empty word and the stack to be empty, and run a clockwise tour of the tree T , starting from the root.
- Every time we encounter a vertex v for the first time, we read its label $\lambda(v)$: if $\lambda(v) = 0$, then we put nothing in the stack and append E to w , else $\lambda(v) > 0$ and we insert (in this order) a letter S and $\lambda(v) - 1$ letters W in the stack and append N to w .
- Every time we encounter a vertex for the last time, we pop the top element of the stack and append it to w .

- The algorithm ends after we reach the root vertex for a second (and last) time.

The word w obtained via this procedure is called the *fish word* of T and we denote it by $\text{FW}(T)$. We provide an example in Figure 3. It is not straightforward that the stack is always not empty when we have to pop an element of the stack, but the conditions on labels of labeled sorting trees ensure this property.

Proposition 3. *The map $\text{FW} : \mathcal{LST} \rightarrow \mathcal{FF}$ is a bijection preserving the size.*

Our proof of the proposition relies on the isomorphic decompositions of labeled sorting trees and of fighting fish (the wasp-waist decomposition presented in [3]) transferred by FW . Combining the maps LST and FW , we then get (up to conjugation) a direct description of the recursive bijection of Fang (see [7]) between two-stack sortable permutations and fighting fish:

Theorem 1. *The map $\text{FW} \circ \text{LST} : 2\mathcal{SS} \rightarrow \mathcal{FF}$ is a bijection that sends two-stack sortable permutations of size n to fighting fish of size n .*

4 Perspectives

- Let $T \in \mathcal{LST}_n$ a labeled sorting tree. Proposition 2 states that there is a unique two-stack sortable permutation σ such that $\text{LST}(\sigma) = T$. It raises the natural question of enumerating the set $\{\tau \in \mathfrak{S}_n, \text{LST}(\tau) = T\}$. An equivalent problem is to determine how many permutations $\tau \in \mathfrak{S}_n$ satisfy $(D_1(\tau), D_2(\tau)) = (D_1(\sigma), D_2(\sigma))$.

- The conditions of the characterization of trees belonging to \mathcal{LST}_n in Proposition 2 do not depend on the order of the children of a given vertex. In particular, the mirror tree (obtained by recursively reversing the order of the children of the nodes) of a tree in \mathcal{LST}_n is also in \mathcal{LST}_n . This symmetry is surprising if we consider that \mathcal{LST}_n is the set of labeled sorting trees of two-stack sortable permutations. On the other hand, it is also not evident that the algorithm defining the bijection FW still produces a valid fighting fish when considering the mirror tree of a tree in \mathcal{LST}_n . It might be interesting to investigate the involutions on two-stack sortable permutations and on fighting fish induced by this mirror involution on labeled sorting trees.

- Similarly, conjugation on fighting fish gives rise to involutions on labeled sorting trees and on two-stack sortable permutations. It would be nice to have a direct description of these involutions, since the counterpart of conjugation on nonseparable planar maps is the important notion of duality (see [4]).

- A natural statistic to consider on fighting fish is the *area*: it is the number of square cells composing it (or equivalently the area enclosed by the corresponding quadrant walk). This statistic has an interesting counterpart on synchronized Tamari intervals (which are bijectively linked to fighting fish in [5]), generalizing the notion of area on Dyck paths. We tried to have a nice and direct interpretation on (two-stack sortable) permutations of the area statistic on fighting fish (transferred by $\text{FW} \circ \text{LST}$), without success. Still, we were able to characterize the permutations giving rise to a fighting fish of minimal area in terms

of pattern avoidance. We also conjecture that they are enumerated according to their size by the sequence [10, A131178].

References

- [1] M. Bousquet-Mélou. Multi-statistic enumeration of two-stack sortable permutations. *The Electronic Journal of Combinatorics*, 5(1):R21, 1998.
- [2] E. Duchi, V. Guerrini, S. Rinaldi, and G. Schaeffer. Fighting Fish. *Journal of Physics A : Mathematical and Theoretical*, 50.2, 2016.
- [3] E. Duchi, V. Guerrini, S. Rinaldi, and G. Schaeffer. Fighting Fish: Enumerative Properties. In *29th International Conference on "Formal Power Series and Algebraic Combinatorics" (FPSAC 2017)*. Séminaire Lotharingien de Combinatoire 78B.43, 2017.
- [4] E. Duchi and C. Henriet. A bijection between rooted planar maps and generalized fighting fish, 2022.
- [5] E. Duchi and C. Henriet. A bijection between Tamari intervals and extended fighting fish. *European Journal of Combinatorics*, 110:103698, 2023.
- [6] S. Dulucq, S. Gire, and O. Guibert. A combinatorial proof of J. West's conjecture. *Discrete Mathematics*, 187(1):71–96, 1998.
- [7] W. Fang. Fighting Fish and Two-Stack Sortable Permutations. In *30th International Conference on "Formal Power Series and Algebraic Combinatorics" (FPSAC 2018)*. Séminaire Lotharingien de Combinatoire 80B.7, 2018.
- [8] I. P. Goulden and J. West. Raney Paths and a Combinatorial Relationship between Rooted Nonseparable Planar Maps and Two-Stack-Sortable Permutations. *Journal of Combinatorial Theory, Series A*, 75(2):220–242, 1996.
- [9] D. E. Knuth. *The Art of Computer Programming, Vol. 1: Fundamental Algorithms*. Addison-Wesley, Reading, Mass., third edition, 1997.
- [10] OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences. Published electronically at <http://oeis.org>.
- [11] J. West. Sorting twice through a stack. *Theoretical Computer Science*, 117(1):303–313, 1993.
- [12] D. Zeilberger. A proof of Julian West's conjecture that the number of two-stack-sortable permutations of length n is $2(3n)!/((n+1)!(2n+1)!)$. *Discrete Mathematics*, 102(1):85–93, 1992.