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A precise condition for independent transversals in bipartite covers

(Extended abstract)

Stijn Cambie* Penny Haxell† Ross J. Kang‡ Ronen Wdowinski§

Abstract

Given a bipartite graph $H = (V = V_A \cup V_B, E)$ in which any vertex in V_A (resp. V_B) has degree at most D_A (resp. D_B), suppose there is a partition of V that is a refinement of the bipartition $V_A \cup V_B$ such that the parts in V_A (resp. V_B) have size at least k_A (resp. k_B). We prove that the condition $D_A/k_A + D_B/k_B \leq 1$ is sufficient for the existence of an independent set of vertices of H that is simultaneously transversal to the partition, and show moreover that this condition is sharp.

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1 Introduction

Consider the following question: how much easier is it to colour graphs that are bipartite than to colour graphs in general? Of course, when considered in the context of the usual chromatic number, this is utterly trivial: compared to the general case, for which the chromatic number can be $\Delta(G) + 1$ but no larger (with $\Delta(G)$ denoting the maximum

*Extremal Combinatorics and Probability Group (ECOPRO), Institute for Basic Science (IBS), Daejeon, South Korea. E-mail: `stijn.cambie@hotmail.com` or `stijncambie@ibs.re.kr`. Supported by the Institute for Basic Science (IBS-R029-C4).

†Department of Combinatorics and Optimization, University of Waterloo, Waterloo, ON Canada N2L 3G1. E-mail: `pehaxell@uwaterloo.ca`. Partially supported by NSERC.

‡Korteweg–de Vries Institute for Mathematics, University of Amsterdam, PO Box 94248, 1090 GE Amsterdam, Netherlands. E-mail: `r.kang@uva.nl`. Partially supported by a Vidi grant (639.032.614) of the Dutch Research Council (NWO).

§Department of Combinatorics and Optimization, University of Waterloo, Waterloo, ON Canada N2L 3G1. E-mail: `ronen.wdowinski@uwaterloo.ca`

degree of G), the factor of reduction in the number of necessary colours is of order $\Delta(G)$. We treat some settings stronger than that of ordinary proper colouring, settings that have both classic and contemporary combinatorial motivation.

Recall the definition of the list chromatic number, a notion introduced nearly half a century ago independently by Erdős, Rubin and Taylor [5] and by Vizing [12]. Let $G = (V, E)$ be a simple, undirected graph. A mapping $L : V(G) \to 2^{\mathbb{Z}^+}$ is called a *list-assignment* of G; if for some positive integer k, the mapping L satisfies $|L(v)| = k$ for all v then it is called a *k-list-assignment*; a colouring $c : V \to \mathbb{Z}^+$ is called an *L-colouring* if $c(v) \in L(v)$ for any $v \in V$. We say G is *k-choosable* if for any k-list-assignment L of G there is a proper L-colouring of G. The *choosability* $\chi_\ell(G)$ (or *choice number* or *list chromatic number*) of G is the least k such that G is k-choosable.

Framing the above question with respect to the list chromatic number, note first that a greedy procedure implies $\chi_\ell(G) \le \Delta(G) + 1$ always, which is exact for G a complete graph. However, for bipartite G, it is a longstanding conjecture that $\chi_\ell(G)$ must be lower than this bound by a factor of order $\Delta(G)/\log \Delta(G)$.

Conjecture 1.1 (Alon and Krivelevich [2]). *There is some $C \ge 1$ such that $\chi_\ell(G) \le C \log_2 \Delta(G)$ for any bipartite graph G with $\Delta(G) \ge 2$.*

If true, this statement would be sharp up to the value of C, due to the complete bipartite graphs [5]. For an idea of how stubborn this problem has been, we relate to the reader how the current best progress was essentially *already known* to the conjecture's originators. In particular, a seminal result for triangle-free graphs of Johansson [8] from the mid-1990's implies that $\chi_\ell(G) = O(\Delta(G)/\log \Delta(G))$ as $\Delta(G) \to \infty$ for any bipartite G, so a reduction factor only of order $\log \Delta(G)$.

To stimulate activity, two of the authors with Alon [1, 4] proposed some natural refinements and variations of Conjecture 1.1, and offered modest related progress. Although less directly relevant to Conjecture 1.1, the present work has the momentum of this trajectory. We introduce some definitions needed to properly describe this progression. In particular, we cast the (bipartite) colouring task in a more precise and general way.

Let G and H be simple, undirected graphs. We say that H is a *cover (graph)* of G with respect to a mapping $L : V(G) \to 2^{V(H)}$ if L induces a partition of $V(H)$ and the bipartite subgraph induced between $L(v)$ and $L(v')$ is edgeless whenever $vv' \notin E(G)$. If for some positive integer k, the mapping L satisfies $|L(v)| = k$ for all v, then we call H a *k-fold* cover of G (with respect to L). Moreover, if G and H are bipartite graphs, where G admits a bipartition $V(G) = A_G \cup B_G$ and H admits a bipartition $V(H) = A_H \cup B_H$, then we say that H is a *bipartite cover (graph)* of H with respect to L if $L(A_G)$ induces a partition of A_H and $L(B_G)$ induces a partition of B_H, i.e. the bipartitions of G and H suitably align. We will have reason to be even more specific for this situation by referring to H as an *(A, B)-cover* of G (with respect to L). (Here we regard A as the pair (A_G, A_H) of partitions, and B similarly.)

To connect the notions above to Conjecture 1.1, notice that, for any list-assignment L of some graph G, one may construct a cover graph H as follows. The vertices of H consist of all pairs (v, x) for $v \in G$ and $x \in L(v)$, and $E(H)$ is a subset of the collection of pairs $(v,x)(v',x')$ such that $vv' \in E(G)$ and $x = x'$. By regarding L as a mapping from v to $\{(v,x) \mid x \in L(v)\}$, we can then regard H as a cover graph of G with respect to L. Moreover, if G is bipartite, the corresponding H is a bipartite cover of G with respect to L. We refer to any (bipartite) cover graph constructed as above as a *(bipartite) list-cover*. Moreover, if $E(H)$ is chosen maximally, we may refer to H as the *maximal (bipartite) list-cover* of G with respect to L. Notice that a proper L-colouring of G is equivalent to an independent set in the corresponding maximal list-cover H that is transversal to the partition induced by L (that is, it intersects every part exactly once), or, in short, an *independent transversal (IT)* of H.

Conjecture 1.1 redux. *There is some $C \geq 1$ such that, for any bipartite graph G of maximum degree $\Delta \geq 2$, any bipartite $(C \log_2 \Delta)$-fold list-cover of G admits an independent transversal.*

There are three potential directions to highlight through adoption of the above notation. First, note that list-covers form a proper subclass of all cover graphs, and so we might consider the 'colouring' task under increasingly more general conditions with respect to H. More specifically, we may ask analogous questions about sufficient conditions for the existence of an IT in natural and successively larger superclasses of list-covers (among all cover graphs). Second, note that if G has maximum degree Δ, then any list-cover of G has maximum degree Δ, but the converse is not true in general. And, for instance, we may consider a problem/result about list-colouring in some class of bounded degree graphs and try to generalise it to the analogous class of bounded degree list-covers. This type of 'colour-degree' problem was introduced by Reed [10]. Third, and specific to (A,B)-covers, we may insist on a more refined viewpoint by imposing (degree/list-size/structural) conditions on parts A and B separately. Two of the authors together with Alon [1] introduced this third asymmetric perspective for studying Conjecture 1.1, and in a follow up [4] they furthermore took on the first two perspectives, in particular by generalising the problem to so-called *correspondence-covers*, which we discuss later. Here we concentrate on the most general case for (asymmetric, bipartite) cover graphs with given degree bounds.

The following problem was posed in [4] (see therein the special case of Problem 1.1 with Δ_A, Δ_B infinite).

Problem 1.2. *Let H be an (A,B)-cover of G with respect to L. What conditions on positive integers k_A, k_B, D_A, D_B suffice to ensure the following? If the maximum degrees in A_H and B_H are D_A and D_B, respectively, and $|L(v)| \geq k_A$ for all $v \in A_G$ and $|L(w)| \geq k_B$ for all $w \in B_G$, then there is guaranteed to be an independent transversal of H with respect to L.*

We resolve Problem 1.2 through the following sufficient condition for a bipartite cover graph to admit an IT.

Theorem 1.3. *Let H be an (A,B)-cover of G with respect to L. Let positive integers k_A, k_B, D_A, D_B be such that $\frac{D_B}{k_A} + \frac{D_A}{k_B} \leq 1$. If the maximum degrees in A_H and B_H are D_A and D_B, respectively, and $|L(v)| \geq k_A$ for all $v \in A_G$ and $|L(w)| \geq k_B$ for all $w \in B_G$, then H admits an independent transversal with respect to L.*

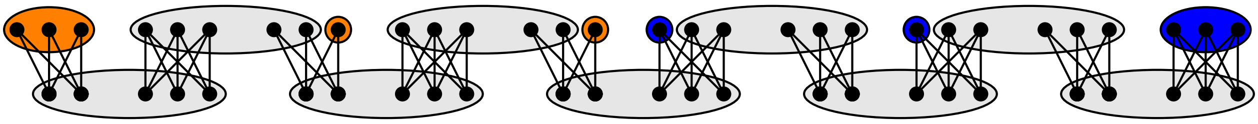

Figure 1: A bipartite graph with maximum degree 3 and partition classes of size 5 with no IT

This result is corollary to a general result for independent transversals found in [6].

In fact, the condition in Theorem 1.3 is best possible, as follows.

Theorem 1.4. *Let positive integers k_A, k_B, D_A, D_B be such that $\frac{D_B}{k_A} + \frac{D_A}{k_B} > 1$. Then there exists an (A, B)-cover H of G with respect to L such that the maximum degrees in A_H and B_H are D_A and D_B, respectively, and $|L(v)| = k_A$ for all $v \in A_G$ and $|L(w)| = k_B$ for all $w \in B_G$, and such that H admits no independent transversal with respect to L.*

It is worth isolating the symmetric situation where we maintain that $D_A = D_B = D$ and $k_A = k_B = k$; in this case the condition in Theorem 1.3 resolves to $k \geq 2D$. In other words, we have the following.

Corollary 1.5. *Any bipartite $(2D)$-fold cover graph of maximum degree D admits an independent transversal. Moreover, the conclusion may fail if the $2D$ part size condition is relaxed to $2D - 1$.*

This condition coincides with that of a well-known, more general result of the second author [7]: that any $(2D)$-fold cover graph of maximum degree D is guaranteed to admit an IT. As such, one may see Theorem 1.4 as simultaneously a strengthening and generalisation of a result of Szabó and Tardos [11] (which in turn built upon a series of results beginning in the original paper of Bollobás, Erdős and Szemerédi [3]): that there exists a $(2D-1)$-fold cover graph of maximum degree D that does not admit an IT. Recalling the question posed at the beginning, Theorem 1.4 shows in a wider sense how the bipartite assumption does *not* help for the existence of ITs in cover graphs.

We remark that, while the construction of Szabó and Tardos is composed of the union of complete bipartite graphs, its partition classes do not align with a bipartition. Corollary 1.5 affirms that it is possible to achieve such an alignment in some bipartite construction. For an indication of the difference, Figure 1 depicts the $D = 3$ construction in Corollary 1.5, and one can compare it with [11, Fig. 1].

Let us briefly discuss what happens in the special case of correspondence-covers, as explored in [4]. Given a cover graph H of G with respect to L, we say H is a *correspondence-cover* if the bipartite subgraph induced between $L(v)$ and $L(v')$ is a matching for any $vv' \in V(G)$. In other words, the maximum degree induced between two parts of H with respect to L is at most 1. Clearly the class of all correspondence-covers strictly includes that of all list-covers. The next result follows from a 'coupon collector' argument, and this is counterbalanced by a simple probabilistic construction (that was given, for example, in [9]).

Theorem 1.6 ([4]). *For any $\varepsilon > 0$, the following holds for all D sufficiently large. Any bipartite $(1+\varepsilon)\frac{D}{\log D}$-fold correspondence-cover graph of maximum degree D admits an independent transversal. Moreover, the conclusion fails if the $(1+\varepsilon)$ factor is weakened to a $(\frac{1}{2}-\varepsilon)$ factor.*

One reason for highlighting this case is that it could be interesting to gradually tune (between 1 and D) the condition on maximum degree induced between two parts of H with respect to L, in order to gain a better understanding of the transition between the $\Theta(D/\log D)$ (probabilistic) part-size condition in Theorem 1.6 and the $\Theta(D)$ condition in Corollary 1.5 (which was originally established in [7]).

Let us conclude by returning to the original motivation and a related challenge. With Corollary 1.5 and Theorem 1.6 in mind, the following 'colour-degree' generalisation of Conjecture 1.1 seems worth investigating.

Conjecture 1.7. *There is some $C \geq 1$ such that any bipartite $(C\log_2 D)$-fold list-cover graph of maximum degree $D \geq 2$ admits an independent transversal.*

To round out the story, we point out how Conjectures 1.1 and 1.7 are essentially equivalent.

Theorem 1.8. *If Conjecture 1.1 is true for some constant $C \geq 1$, then Conjecture 1.7 is true for some constant $C' \geq 1$. The same implication holds when C and C' are both replaced by $1+o(1)$ (as $\Delta, D \to \infty$).*

Proof. Assume Conjecture 1.1 is true for some $C \geq 1$. We choose $D_0 \geq 2$ such that $\sqrt{D} \geq C\log_2 D$ for every $D \geq D_0$, and take $C' = 2D_0 \geq 2C^2 \geq 2C$. We will prove that any bipartite $(C'\log_2 D)$-fold list-cover graph of maximum degree D admits an independent transversal. Let $k = C'\log_2 D$ and H be a k-fold list-cover of maximum degree $D \geq 2$. If $D \leq D_0$, then $k \geq C' \geq 2D$ and it then follows from Haxell's theorem [7] that H has an independent transversal as desired. We may therefore assume $D > D_0$. Let G be the 'covered' graph of H, i.e. uv is an edge of G if and only if $L(u) \cap L(v)$ is not empty. Then by definition the maximum degree Δ of G satisfies $D_0 \leq D \leq \Delta \leq kD$. By the choice of D_0 it then follows that $\sqrt{\Delta} \geq C\log_2 \Delta$. Suppose now for a contradiction that $k < C\log_2 \Delta$. Then $\Delta < D \cdot C\log_2 \Delta$ and so $D > \frac{\Delta}{C\log_2 \Delta} \geq \sqrt{\Delta}$. But $D > \sqrt{\Delta}$ and $C' \geq 2C$ imply that $k = C'\log_2 D \geq C\log_2 \Delta$, which is a contradiction. Hence $k \geq C\log_2 \Delta$. Now consider the maximal list-cover $H' \supseteq H$ of G with respect to L. By our assumption, H' admits an independent transversal with respect to L, which implies the same conclusion for H, as required.

The proof for the $1+o(1)$ version proceeds analogously. Fix $\varepsilon > 0$. Now one can take D_0 sufficiently large such that $C\log_2 D \leq D^{\varepsilon}$ and Conjecture 1.1 is true with $1+\varepsilon$ whenever $\Delta \geq D_0$. Then for any k-fold list-cover H of maximum degree $D \geq 2$, where $k \geq \frac{1+\varepsilon}{1-\varepsilon}\log_2 D$, we conclude H has an independent transversal by the same strategy. □

In a similar way, nontrivial progress on Conjecture 1.1 may imply nontrivial progress on Conjecture 1.7. Conversely, lower bound constructions related to Conjecture 1.7 may directly yield corresponding constructions related to Conjecture 1.1.

2 A sufficient condition

In this section, we derive Theorem 1.3.

We say that a set U of vertices of a graph G *dominates* the set $W \subseteq V(G)$ if every vertex of W has a neighbour in U. (This is a somewhat nonstandard use of the term since, contrary to the most common usage, here we require each vertex of $U \cap W$ to have a neighbour in U.) Theorem 1.3 is a straightforward consequence of the following result of Haxell (see e.g. [6]), concerning critical graphs with respect to ITs.

Theorem 2.1 ([6])**.** *Let $H = (V_H, E_H)$ be a cover graph of some graph $G = (V_G, E_G)$ with respect to L. Suppose that H has no IT but $H - e$ does for any $e \in E_H$. Then for any $e \in E_H$, there exists a subset $S \subset V_G$ and a set Z of edges of the subgraph of H induced by $L(S)$ such that $e \in Z$, $|Z| \le |S| - 1$, and $V_H(Z)$ dominates $L(S)$.*

Proof of Theorem 1.3. Suppose H is a counterexample and take it to be edge-minimal. By Theorem 2.1, there exist some a partition classes of A_H and b partition classes of B_H, and a set Z of edges of size at most $a + b - 1$ whose end-vertices dominate the union of these $a + b$ partition classes. The end-vertices of Z dominate at most $(a + b - 1)D_B$ vertices in A_H, while the a partition classes contain at least ak_A vertices. This implies that $ak_A \le (a + b - 1)D_B$. Similarly, considering B_H, we have $bk_B \le (a + b - 1)D_A$. But then

$$\frac{D_A}{k_B} + \frac{D_B}{k_A} \ge \frac{b}{a+b-1} + \frac{a}{a+b-1} > 1,$$

contradicting the hypothesis. □

3 Sharpness of the condition

A proof of Theorem 1.4 is deferred to the full manuscript associated to this extended abstract.

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