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Countable ultrahomogeneous 2-colored graphs consisting of disjoint unions of cliques

(Extended abstract)

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Abstract

We classify the countable ultrahomogeneous 2-vertex-colored graphs in which the color classes form disjoint unions of cliques. This generalizes work by Jenkinson et. al. [8], Lockett and Truss [11] as well as Rose [13] on ultrahomogeneous n-graphs. As the key aspect in such a classification, we identify a concept called piecewise ultrahomogeneity. We prove that there are two specific graphs whose occurrence essentially dictates whether a graph is piecewise ultrahomogeneous, and we exploit this fact to prove the classification.

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1 Introduction

Ultrahomogeneous structures are relational structures in which every isomorphism between finite substructures can be extended to an automorphism of the entire structure.[1] The extensive study of ultrahomogeneous objects relates various areas of research, such as model theory, permutation group theory and Ramsey theory (see [12] for a survey). A vast collection of ultrahomogeneous classes of relational structures has been classified.

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[1]Some authors use the term "homogeneous" for this property.

For instance, apart from different classes of graphs, which we discuss below, there exist classification results for partially ordered sets [14], tournaments [9, 2] as well as countably infinite permutations [1].

In this article, we focus on a special class of countable ultrahomogeneous graphs. By work of Sheehan [15] and Gardiner [3] as well as Golfand and Klin [4], the finite ultrahomogeneous graphs are known. Lachlan and Woodrow [10] gave a characterization of the ultrahomogeneous graphs with countably infinitely many vertices. Cherlin [2] asked for a classification of ultrahomogeneous n-graphs, that is, ultrahomogeneous graphs for which the vertex set is partitioned into n subsets which are respected by the partial isomorphisms considered.

Nowadays, one usually thinks of n-graphs as graphs with a vertex-coloring in n colors, and considers isomorphisms preserving colors. Finite ultrahomogeneous vertex-colored graphs were classified in [5]. Every color class in an ultrahomogeneous graph induces a monochromatic ultrahomogeneous graph. In particular, every infinite color class forms an independent set or a disjoint union of cliques, or it induces a Rado graph or a Henson graph (see [10]). Jenkinson et. al. [8] considered vertex-colored graphs in which the color classes form independent sets. Their work was extended by Lockett and Truss [11] who allowed an additional coloring of the edges (while still requiring that every color class forms an independent set). In his dissertation, Rose [13] investigates countable 2-colored graphs. The main part of his work covers the case that one color class forms a disjoint union of cliques and the other one induces a Rado graph or a Henson graph. For the case that both color classes form disjoint unions of cliques, a partial list of possible cases is stated, but not proven.

In this paper, we classify the countable 2-colored ultrahomogeneous graphs for which both color classes form disjoint unions of cliques. We identify a new concept, which we call piecewise ultrahomogeneity, as key aspect in such classifications. An ultrahomogeneous graph whose color classes form disjoint unions of cliques is called piecewise ultrahomogeneous if each subgraph induced by a pair of maximal cliques of distinct color is ultrahomogeneous. As explained by Rose (see [13, Theorem 5.2]), this concept also appears in the dissertation of Jenkinson [7]. We obtain the following characterization of piecewise ultrahomogeneity (see Theorems 4.1 and 5.2):

Theorem A. *Let G be a non-bipartite, countable, 2-colored ultrahomogeneous graph in which the color classes form disjoint unions of cliques and that is not a blow-up. Apart from one degenerate case $F_{2,2}$, the graph G is piecewise ultrahomogeneous if and only if it contains induced subgraphs isomorphic to the graphs Q and $\widetilde{Q}$ depicted in Figure 1.*

We leverage the theorem to completely classify countable 2-colored ultrahomogeneous graphs in which the color classes form disjoint unions of cliques:

Theorem B. *Let G be a countable 2-colored ultrahomogeneous graph in which the color classes form disjoint unions of cliques and that is not a blow-up. Then (after possibly interchanging the colors) exactly one of the following holds:*

(i) *(Piecewise ultrahomogeneous, Theorem 6.1) Either both color classes in G form an independent set or a single clique, G belongs to a single biparametric family* $\{G_{r,b}\colon r, b \in \mathbb{N} \cup \{\infty\}\}$*, or G is isomorphic to the specific graph* $F_{2,2}$*.*

(ii) *(Not piecewise ultrahomogeneous, Theorem 5.2) The graph G belongs to one of two monoparametric families* $\{F^k_{\infty,1}\colon k \in \mathbb{N}_{\geq 2}\}$ *or* $\{F^k_{\infty,2}\colon k \in \mathbb{N}_{\geq 2}\}$*, or it is isomorphic to one of four specific graphs* $F_{2,1}$, $F_{\infty,1}$, $F_{\infty,2}$ *or* $F_{\infty,\infty}$*.*

This paper is organized as follows: Section 2 contains preliminary results. In Section 3, we recall Fraïssé's theory and study the structure of minimally omitted subgraphs. In Section 4, we introduce the concept of piecewise ultrahomogeneity and prove one implication of Theorem A. In Sections 5 and 6, we classify graphs that are not piecewise ultrahomogeneous and piecewise ultrahomogeneous, respectively, thereby proving Theorems A and B. We conclude with some final remarks in Section 7.

2 Preliminaries

Let G be a (simple) graph. We denote by $V(G)$ the vertex set of G. A *2-colored graph* is a graph whose vertices are colored in two distinct colors, which we usually call "blue" and "red". The graph G is *ultrahomogeneous* if every partial isomorphism between two finite induced subgraphs of G extends to an automorphism of G. In order to shorten our notation, we call a graph G *clique-ultrahomogeneous* (CUH) if G is a countably infinite ultrahomogeneous graph on blue and red vertices such that both color classes are disjoint unions of cliques. Two distinct vertices $v, v' \in V(G)$ are called *twins* if they have the same color and the same neighbors in $G - \{v, v'\}$. The edges in G with endpoints of different color are called *cross edges*. We write $\widetilde{G}$ for the graph obtained from G by complementing the cross edges while maintaining the edges within each color class. Let $\mathcal{R}$ and $\mathcal{B}$ denote the sets of maximal red and blue cliques of G, respectively. By [10], the elements of $\mathcal{R}$ all have the same size (similarly for $\mathcal{B}$). Note that the automorphism group of G permutes the set $\mathcal{R}$. Similarly, it permutes $\mathcal{B}$. From the definition of ultrahomogeneity, we obtain the following statement (also see [5, Lemma 6.1]):

Lemma 2.1. *Let G be a 2-colored ultrahomogeneous graph, and let H be obtained from G by any combination of complementations of the edges within a color class or the cross edges. Then H is ultrahomogeneous.*

Let H be a 2-colored graph in which one color class is an independent set. We call G a *blow-up* of H if G is obtained from H by, for some $i \in \mathbb{N}_{\geq 2} \cup \{\infty\}$, replacing all vertices in this color class by i-cliques and joining their vertices to the neighbors of the original vertex in H. The following property is easily verified (also see [5, Lemma 6.2]):

Lemma 2.2. *A blow-up of a graph H is ultrahomogeneous if and only if H is ultrahomogeneous.*

We call a CUH graph *basic* if $|\mathcal{R}|, |\mathcal{B}| \geq 2$ holds, and G is not a blow-up. By complementation inside the color classes and reduction of blow-ups, which preserves ultrahomogeneity (see Lemmas 2.1 and 2.2), we can always pass to a basic CUH graph. It therefore suffices to consider basic graphs. A 2-colored graph G is called *trivial* if all or none of the possible cross edges in G are present. Concerning the sizes of the color classes, we note the following:

Lemma 2.3. *Let G be a basic CUH graph. If a color class of G is finite, then G is trivial.*

By [8], there exists a unique countably infinite 2-colored ultrahomogeneous graph whose color classes form independent sets and which is generic in the following sense: For every $c \in \{\text{red}, \text{blue}\}$ and all finite disjoint subsets S, T of color c, there exists a vertex of color $c' \neq c$ adjacent to all vertices in S and to none of the vertices in T. This graph is called the *generic bipartite graph* (GB-graph). We frequently use the following classification:

Theorem 2.4 ([8, Theorem 2.2])**.** *Let G be a countable 2-colored ultrahomogeneous graph whose color classes form independent sets. Either G is trivial, the cross edges in G form a perfect matching or its complement, or G is isomorphic to the GB-graph.*

Note that the graphs given in Theorem 2.4 are bipartite.

3 Fraïssé limits and omitted subgraphs

Let L be a countable relational language. An L-structure D is called *ultrahomogeneous* if every isomorphism between finite substructures of D extends to an automorphism of D. The *age* of an L-structure D is the class of all finite L-structures that are isomorphic to induced substructures of D. An *amalgamation class* is a class of finite L-structures which is closed under isomorphism and taking induced substructures, and has the *amalgamation property*: For $J, A_1, A_2 \in \mathcal{A}$ and embeddings $\iota_i \colon J \to A_i$ $(i = 1, 2)$, there exists $A \in \mathcal{A}$ and embeddings $\kappa_i \colon A_i \to A$ $(i = 1, 2)$ such that $\kappa_1 \circ \iota_1 = \kappa_2 \circ \iota_2$ holds. Then A is called an *amalgam* of A_1 and A_2.

Theorem 3.1 (Fraïssé)**.** *Let D be a countable ultrahomogeneous L-structure. Then the age of D is an amalgamation class. Conversely, for every amalgamation class $\mathcal{C}$ of finite L-structures, there exists a countable ultrahomogeneous L-structure D with age $\mathcal{C}$, and D is unique up to isomorphism.*

In the setting of Theorem 3.1, we call D the *Fraïssé limit* of $\mathcal{C}$. Further information on this topic can be found, for example, in [6]. We now return to the special case of countable 2-colored graphs. If H is a finite graph that is isomorphic to an induced subgraph of a graph G, we say that H is *realized* in G. Otherwise, H is *omitted* in G. The graph H is called *minimally omitted* if H is omitted in G and every proper induced subgraph of H is realized in G. The set of minimally omitted subgraphs of G is denoted by $O(G)$. Concerning the structure of the color classes of a graph in $O(G)$, we obtain the following result, which forms the basis for all further arguments:

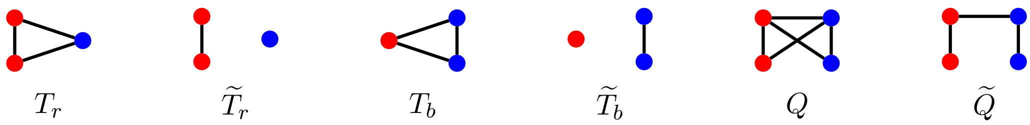

Figure 1: The graphs in $\mathcal{T} \cup \{Q, \widetilde{Q}\}$

Theorem 3.2. *Let G be a CUH graph. If $H \in O(G)$ is not monochromatic, then for every color $c \in \{red, blue\}$, one of the following holds:*

(i) The vertices of color c in H form a clique of size at least 3 and they are all twins in H,

(ii) the vertices of color c in H form a 2-clique, or,

(iii) the vertices of color c in H form an independent set.

4 Piecewise ultrahomogeneity

We call a CUH graph G *piecewise ultrahomogeneous* if for every $R \in \mathcal{R}$ and $B \in \mathcal{B}$, the graph $G[R \cup B]$ is ultrahomogeneous. Let T_r and T_b be the triangles containing a single blue vertex and a single red vertex, respectively. We set $\mathcal{T} = \{T_r, \widetilde{T}_r, T_b, \widetilde{T}_b\}$. Moreover, let Q be the graph arising from a complete graph on two red and two blue vertices by omitting one cross edge. The graphs in $\mathcal{T}$ as well as Q and $\widetilde{Q}$ are depicted in Figure 1.

Theorem 4.1. *Let G be a basic CUH graph. If Q and $\widetilde{Q}$ are realized in G, then G is piecewise ultrahomogeneous.*

Proof (Sketch). Let $R \in \mathcal{R}$ and $B \in \mathcal{B}$. We show that $G[R \cup B]$ is the complement of the GB-graph. To this end, consider finite disjoint subsets S and T of R. We need to show that there exists a vertex $v \in B$ adjacent to all vertices in S and to none of the vertices in T. The vertices in $S \cup T$ form a red clique of size $|S| + |T|$. In a series of lemmas, we show that there exists a vertex $v' \in B$ whose neighborhood in $S \cup T$ is a set S' of size $|S|$. Here, the main technical difficulty is to ensure that the blue vertex with the required neighborhood can be found in the clique B. Moreover, we prove that there exists a joint neighbor $b \in B$ of the vertices in $S \cup T$. Now consider the partial isomorphism φ of G obtained by bijectively mapping S' to S and $(S \cup T) \setminus S'$ to T while fixing b. Let $\hat{\varphi}$ be an automorphism of G extending φ. Then $\hat{\varphi}(v) \in B$ is a vertex adjacent to all vertices in S and to none of the vertices in T. For the other color class, one can argue similarly. Hence $G[R \cup B]$ is ultrahomogeneous. □

5 CUH graphs omitting Q or $\widetilde{Q}$

In this section, we classify the basic CUH graphs which omit Q or $\widetilde{Q}$. We first determine the possible graphs in $O(G)$.

Theorem 5.1. *Let G be a basic CUH graph in which Q or $\widetilde{Q}$ is omitted, and which is not isomorphic to one of the graphs in Theorem 2.4. Then every non-monochromatic graph in $O(G)$ is contained in $\mathcal{T} \cup \{Q, \widetilde{Q}\}$. Moreover, we have $T_r \in O(G)$ if and only if $\widetilde{T}_r \in O(G)$ holds, and the same is true for T_b and $\widetilde{T}_b$ as well as Q and $\widetilde{Q}$.*

Proof (Sketch). One first shows that $T_r \in O(G)$ holds if and only if $\widetilde{T}_r \in O(G)$ holds, and that this is the case precisely if the maximal red cliques in G have size 2 (similarly for the blue color class). Moreover, we show that if neither the blue nor the red vertices in G form an independent set, then for every $R \in \mathcal{R}$ and $B \in \mathcal{B}$, there exist partitions $R = R_1 \dot\cup R_2$ and $B = B_1 \dot\cup B_2$ such that $(R_1 \times B_1) \cup (R_2 \times B_2)$ is precisely the set of cross edges between R and B. Using this structure, we show that the maximal monochromatic cliques in G either have size 1 or 2, or they are infinite.

Let $H \in O(G)$ and assume that H is not monochromatic. Using Theorem 3.2, we show that either $H \in \mathcal{T} \cup \{Q, \widetilde{Q}\}$ holds, or that the color classes in H form independent sets of size at least 2 (all other possibilities can be eliminated by using the structure of $G[R \cup B]$ described above). We then show that the second case cannot occur. The main technical difficulty is to show that for a given monochromatic independent set $J \subseteq V(G)$ of color $c \in \{\text{red, blue}\}$ and a vertex v of color $c' \neq c$, there exist sufficiently many maximal cliques of color c' containing vertices with the same neighbors in J as v. Using this property, we can then successively show that G contains a subgraph isomorphic to H. □

Let $\mathcal{C}$ be the class of finite graphs on red and blue vertices whose color classes form disjoint unions of cliques.

Theorem 5.2. *Let G be a basic CUH graph that omits Q or $\widetilde{Q}$, and that is not isomorphic to one of the graphs in Theorem 2.4. Up to exchanging the color classes, one of the following cases arises:*

(i) If $k := |\mathcal{R}|$ is finite, then G is isomorphic to one of the following graphs:

- *(a) The Fraïssé limit $F^k_{\infty,1}$ of the class of graphs in $\mathcal{C}$ that omit a red independent set of size $k+1$ and the blue 2-clique.*
- *(b) The Fraïssé limit $F^k_{\infty,2}$ of the class of graphs in $\mathcal{C}$ that omit a red independent set of size $k+1$, the blue triangle as well as T_b and $\widetilde{T}_b$.*

(ii) Otherwise, G is isomorphic to one of the following graphs:

- *(a) The Fraïssé limit $F_{2,1}$ of the class of graphs in $\mathcal{C}$ that omit the red triangle, the blue 2-clique, T_r and $\widetilde{T}_r$.*
- *(b) The Fraïssé limit $F_{\infty,1}$ of the class of graphs in $\mathcal{C}$ that omit the blue 2-clique.*
- *(c) The Fraïssé limit $F_{2,2}$ of the class of graphs in $\mathcal{C}$ that omit monochromatic triangles and the graphs in $\mathcal{T}$.*

(d) *The Fraïssé limit* $F_{\infty,2}$ *of the class of graphs in* $\mathcal{C}$ *that omit the blue triangle as well as* T_b *and* $\widetilde{T}_b$*.*

(e) *The Fraïssé limit* $F_{\infty,\infty}$ *of the class of graphs in* $\mathcal{C}$ *that omit* Q *and* $\widetilde{Q}$*.*

Proof (Sketch). One needs to verify that the given graph classes fulfill the amalgamation property. Proving that the above list is exhaustive is done by using Theorem 5.1. □

We remark that the graph $F_{2,1}$ given in Theorem 5.2 appears to be excluded by the (unproven) enumeration of possible CUH graphs stated in [13]. Combining Theorems 4.1 and 5.2 yields the characterization of piecewise ultrahomogeneity stated in Theorem A.

6 Classification of piecewise ultrahomogeneous graphs

Using the results in Theorem 2.4, we obtain the following classification:

Theorem 6.1. *Let* G *be a basic piecewise ultrahomogeneous CUH graph. Then* G *is isomorphic to one of the graphs in Theorem 2.4, to the specific graph* $F_{2,2}$*, or to the Fraïssé limit* $G_{|\mathcal{R}|,|\mathcal{B}|}$ *of the class* $\mathcal{A}_{|\mathcal{R}|,|\mathcal{B}|}$ *of finite graphs on red and blue vertices in which the red and the blue color class form disjoint unions of at most* $|\mathcal{R}|$ *and* $|\mathcal{B}|$ *cliques, respectively.*

For the proof, one first proves that if G is neither isomorphic to one of the graphs in Theorem 2.4 nor to $F_{2,2}$, then $G[R \cup B]$ is the complement of the GB-graph for every $R \in \mathcal{R}$ and $B \in \mathcal{B}$. Using the structure of the GB-graph, one then shows that every graph in $\mathcal{A}_{|\mathcal{R}|,|\mathcal{B}|}$ is realized in G. By Fraïssé's theorem, this implies $G \cong G_{|\mathcal{R}|,|\mathcal{B}|}$. This completes the classification given in Theorem B.

7 Conclusion

In this paper, we classified the countable 2-colored ultrahomogeneous graphs in which each color class forms a disjoint union of cliques. Our key tool was the concept of piecewise ultrahomogeneity introduced in Section 4. We showed that with one exception, a basic non-bipartite CUH graph is piecewise ultrahomogeneous if and only if two specific graphs appear as induced subgraphs (see Theorem A). Using this result, we obtained the classification of countable 2-colored CUH graphs given in Theorem B.

There are several natural continuations of this paper. For example, it would be interesting to classify edge-colored versions of CUH graphs, extending the work of Lockett and Truss [11]. Moreover, one could investigate n-colored versions of CUH graphs for an arbitrary number $n \in \mathbb{N}$. In both cases, we believe that a suitable generalization of piecewise ultrahomogeneity could play a central role. Just as in the case studied in this paper, one could hope to characterize the piecewise ultrahomogeneous graphs in terms of a small number of induced subgraphs, and then use the classifications of ultrahomogeneous multipartite graphs given in [8] and [11]. Conversely, if a graph fails to be piecewise ultrahomogeneous, its structure might again be very limited.

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