Single-conflict colorings of degenerate graphs

(Extended abstract)

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Abstract

We consider the *single-conflict coloring* problem, in which each edge of a graph receives a forbidden ordered color pair. The task is to find a vertex coloring such that no two adjacent vertices receive a pair of colors forbidden at an edge joining them. We show that for any assignment of forbidden color pairs to the edges of a d-degenerate graph G on n vertices of edge-multiplicity at most $\log \log n$, $O(\sqrt{d} \log n)$ colors are always enough to color the vertices of G in a way that avoids every forbidden color pair. This answers a question of Dvořák, Esperet, Kang, and Ozeki for simple graphs (Journal of Graph Theory 2021).

1 Introduction

We consider graphs without loops and possibly with parallel edges. A *coloring* of a graph G is a function $\phi : V(G) \to C$ that assigns a color from some color set C to each vertex of

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G. In this paper, we consider a special version of graph colorings known as *single-conflict colorings*, defined as follows. Let G be a graph, and let C be a color set. Let f be a function such that for each edge $e \in E(G)$ with endpoints u and v, f maps the triple (u, e, v) to a forbidden color pair (c_1, c_2), and f maps the triple (v, e, u) to the reverse forbidden color pair (c_2, c_1). Then, we say that a (not necessarily proper) coloring $\phi : V(G) \to C$ is a single-conflict coloring with respect to f and C if $f(u, e, v) \neq (\phi(u), \phi(v))$ for each edge $e = uv$ of G. We call the image of a triple (u, e, v) under f a *conflict*, and we call f a *conflict function*. If k is the minimum integer for which a graph G always has a single-conflict coloring for any color set C of size k and any conflict function f, then we say that k is the *single-conflict chromatic number* of G, and we write $\chi_{\leftrightarrow}(G) = k$.

1.1 Background

Dvořák and Postle [5] first introduced a concept similar to single-conflict coloring, called DP-coloring. Independently, single-conflict coloring was considered by Fraigniaud, Heinrich, and Kosowski [6][1], and the notion of single-conflict chromatic number was later introduced by Dvořák, Esperet, Kang, and Ozeki [4]. In [4], the authors proved the following:

Theorem 1.1 ([4]). *If G is a graph of maximum degree Δ, then $\chi_{\leftrightarrow}(G) \leq \left\lceil \sqrt{e(2\Delta - 1)} \right\rceil$.*

In fact, a stronger bound than in Theorem 1.1 with the leading multiplicative constant of 2 (instead of $\sqrt{2e}$) can be shown. Surprisingly, the factor of 2 is asymptotically sharp, as shown very recently by Groenland, Kaiser, Treffers, Wales [8]. For simple graphs, an even better coefficient of $1 + o(1)$ holds, which can be derived from a result of Kang and Kelly [11], or from an independent result Glock and Sudakov [7]. The last three cited results were stated for independent transversals. Another noteworthy result concerning the single-conflict chromatic number of graphs on surfaces was shown in [4].

Theorem 1.2 ([4]). *If G is a simple graph of Euler genus g, then $\chi_{\leftrightarrow}(G) = O((g + 1)^{1/4} \log(g + 2))$.*

Furthermore, the authors of [4] show that a graph of average degree $\overline{d}$ has a single-conflict chromatic number of at least $\left\lfloor \sqrt{\frac{\overline{d}}{\log \overline{d}}} \right\rfloor$.

The notion of a single-conflict coloring is a generalization of several graph coloring variants. Most immediately, single-conflict colorings generalize the notion of proper colorings as follows. Given a graph G, let $G^{(k)}$ denote the graph obtained from G by replacing each edge of G with k parallel edges. Then, $\chi(G) \leq k$ if and only if $G^{(k)}$ has a single-conflict coloring with a set C of k colors when each set of k parallel edges in $G^{(k)}$ is assigned k distinct monochromatic conflicts.

A single-conflict coloring is also a generalization of a *DP-coloring*, first introduced by Dvořák and Postle [5] under the name of *correspondence coloring*. One may define a DP-coloring of a graph G as a single-conflict coloring of a graph G' on $V(G)$ which is obtained

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[1]Although [6] was published before [5], the arXiv version of [5] appears approximately two months before that of [6].

as follows. First, for each edge $uv \in E(G)$, select a matching M_{uv} in the complete bipartite graph $C \times C$. Then, for each edge $(c_1, c_2) \in M_{uv}$, give G' an edge e with endpoints u, v and a conflict $f(u, e, v) = (c_1, c_2)$. In this way, the single-conflict coloring problem can represent every instance of a DP-coloring problem.

Furthermore, a single-conflict coloring is a generalization of an earlier concept known as an *adapted coloring*, introduced by Hell and Zhu [10], which is defined as follows. Given a graph G with a (not necessarily proper) edge coloring ψ, an *adapted coloring* on G is a (not necessarily proper) vertex coloring ϕ of G in which no edge e is colored the same color as both of its endpoints u and v—that is, $\neg\,(\psi(e) = \phi(u) = \phi(v))$. In other words, if $e \in E(G)$ is a `red` edge, then both endpoints of e may not be colored `red`, but both endpoints of e may be colored, say, `blue`, and the endpoints of e may also be colored with two different colors. The *adaptable chromatic number* of G, written $\chi_{ad}(G)$, is the minimum integer m such that for any edge coloring of G using a set C of m colors, there exists an adapted vertex coloring of G using colors of C. It is easy to see that an adapted coloring on G is a single-conflict coloring on G when each edge $e \in E(G)$ is assigned the monochromatic conflict $(\psi(e), \psi(e))$. Therefore, for every graph G, $\chi_{ad}(G) \leq \chi_{\leftrightarrow}(G)$.

For graphs G of maximum degree Δ, Molloy and Thron [15] show that $\chi_{ad}(G) \leq (1 + o(1))\sqrt{\Delta}$. Molloy [13] shows furthermore that graphs G with chromatic number $\chi(G)$ satisfy $\chi_{ad}(G) \geq (1 + o(1))\sqrt{\chi(G)}$, implying that $\sqrt{\chi(G)}$, $\chi_{ad}(G)$, $\chi_{\leftrightarrow}(G)$, and $\sqrt{\Delta}$ all only differ by a constant factor for graphs G satisfying $\chi(G) = \Theta(\Delta)$. The parameters $\sqrt{\chi(G)}$, $\chi_{ad}(G)$, and $\chi_{\leftrightarrow}(G)$ can also differ by a constant factor even when $\chi(G)$ is not of the form $\Theta(\Delta)$. For instance, for graphs G of maximum degree Δ without cycles of length 3 or 4, the parameters $\sqrt{\chi(G)}$, $\chi_{\leftrightarrow}(G)$, and $\chi_{ad}(G)$ are all of the form $O\left(\sqrt{\frac{\Delta}{\log \Delta}}\right)$ [2, 14], and since randomly constructed Δ-regular graphs of girth 5 often have chromatic number as high as $\frac{1}{2}\frac{\Delta}{\log \Delta}$ [14], this upper bound is often tight.

We note that adapted colorings are equivalent to the notion of *cooperative colorings*, which are defined as follows. Given a family $\mathcal{G} = \{G_1, \ldots, G_k\}$ of graphs on a common vertex set V, a cooperative coloring on $\mathcal{G}$ is defined as a family of sets $R_1, \ldots, R_k \subseteq V$ such that for each $1 \leq i \leq k$, R_i is an independent set of G_i, and $V = \bigcup_{i=1}^{k} R_i$. A cooperative coloring problem may be translated into an adapted coloring problem by coloring the edges of each graph $G_i \in \mathcal{G}$ with the color i and then considering the union of all graphs in $\mathcal{G}$. Overall, this gives us the following observation.

Observation 1.3. *Given a family $\mathcal{G} = \{G_1, \ldots, G_k\}$ of graphs on a common vertex set, the cooperative coloring problem on $\mathcal{G}$ is equivalent to the adapted coloring problem on the edge-colored graph $G = \bigcup_{i=1}^{k} G_i$ in which each edge originally from G_i is colored with the color i.*

It is straightforward to show that Theorem 1.1 implies that a graph family $\mathcal{G}$ containing k graphs of maximum degree Δ on a common vertex set V has a cooperative coloring whenever $k \geq 2e\Delta$. In fact, Haxell [9] showed earlier that it is sufficient to let $k \geq 2\Delta$.

1.2 Our results

We have seen that for graphs G of maximum degree Δ, $\chi_{\leftrightarrow}(G) = O(\sqrt{\Delta})$. However, it is natural to ask whether we can obtain a better upper bound when G has bounded degeneracy. For example, in the related problem of cooperative coloring, Aharoni, Berger, Chudnovsky, Havet, and Jiang [1] obtained the following improved result by considering families of 1-degenerate graphs, i.e. forests:

Theorem 1.4 ([1]). *If $\mathcal{T}$ is a family of forests of maximum degree Δ on a common vertex set V, then there exists a value $k = (1 + o(1)) \log_{4/3} \Delta$ such that if $|\mathcal{T}| \geq k$, then $\mathcal{T}$ has a cooperative coloring.*

One key tool used to prove Theorem 1.4 is an application of the Lovász Local Lemma in which each vertex $v \in V$ receives a random *inventory* S_v of colors from a color set C indexing the forests in $\mathcal{T}$, and then a color c is deleted from S_v if c also belongs to the inventory S_w of the parent w of v in the forest of $\mathcal{T}$ indexed by c. Dvořák, Esperet, Kang, and Ozeki [4] also posed the following question, asking whether this upper bound can be improved for certain degenerate graphs.

Question 1.5. *Suppose G is a d-degenerate graph on n vertices. Is it true that $\chi_{\leftrightarrow}(G) = O(\sqrt{d} \log n)$?*

The authors remarked that a positive answer to Question 1.5 would give an alternative proof of Theorem 1.2. In this paper, we will prove the following theorem, which shows the upper bound of $\chi_{\leftrightarrow}(G) = O(\sqrt{\Delta})$ can often be improved for graphs of bounded degeneracy.

Theorem 1.6. *If G is a d-degenerate graph with maximum degree Δ and edge-multiplicity at most μ, then*
$$\chi_{\leftrightarrow}(G) \leq \left\lceil \sqrt{d} \cdot 2^{\mu/2+2} \sqrt{\mu} \sqrt{1 + \log((d+1)\Delta)} \right\rceil.$$

Theorem 1.6 gives a large class of d-degenerate graphs G satisfying $\chi_{\leftrightarrow}(G) = O(d^{\frac{1}{2}+o(1)})$, containing in particular those d-degenerate simple graphs G with maximum degree $\Delta = \exp(d^{o(1)})$. This upper bound is close to best possible, since Molloy [13] shows that d-degenerate graphs G of chromatic number $d+1$ satisfy $\chi_{\leftrightarrow}(G) \geq \chi_{ad}(G) \geq (1+o(1))\sqrt{d+1}$. By applying the argument used for Theorem 1.6 to simple graphs, we also obtain the following theorem.

Theorem 1.7. *If G is a d-degenerate simple graph of maximum degree Δ, then*
$$\chi_{\leftrightarrow}(G) \leq \left\lceil 2\sqrt{d \left[1 + \log((d+1)\Delta)\right]} \right\rceil.$$

Theorem 1.7 immediately answers Question 1.5 for simple graphs and thus also implies Theorem 1.2. In fact, the result that we will prove is slightly stronger than Theorem 1.7, and we will obtain the following corollary, which generalizes Theorem 1.4 at the expense of a constant factor.

Corollary 1.8. *Let $\mathcal{G}$ be a family of k graphs on a common vertex set V. Suppose each graph $G \in \mathcal{G}$ is at most d-degenerate and of maximum degree Δ. Then, whenever $k \geq 13(1 + d \log(d\Delta))$, $\mathcal{G}$ has a cooperative coloring.*

One natural question is whether the logarithmic factors are necessary in these upper bounds. While we are unable to answer these questions exactly, we note that an upper bound of less than $d + 1$ is unachievable, as Kostochka and Zhu [12] give examples of d-degenerate graphs G that satisfy $\chi_{ad}(G) = d + 1$. Additionally, Question 1.5 remains open for graphs of large edge-multiplicity.

2 Uniquely restrictive conflicts

It is well known that an oriented graph with maximum out-degree d is d-degenerate. Therefore, rather than working directly with d-degenerate graphs, we will consider the larger class of oriented graphs of maximum out-degree d. Given an oriented graph G, we write $A(G)$ for the set of arcs of G. For a vertex $v \in V(G)$, we write $A^+(v)$ for the set of arcs outgoing from v, and we write $A^-(v)$ for the set of arcs incoming to v. Given an arc $e = uv$ in an oriented graph G, and given a conflict function f on G, we will often write $f(e) = f(u, e, v)$.

Consider a color set C and an oriented graph G with a conflict function f. First, given a vertex $v \in V(G)$ and an arc $e \in A(G)$ containing v and second endpoint u, we say that the (v, e) *conflict color* is the first color appearing in the ordered pair $f(v, e, u)$. We write $\mathrm{cc}(v, e)$ for the (v, e) conflict color. Then, we have the following definition.

Definition 2.1. Let $w \in V(G)$. Suppose that for each parallel arc pair $e_1, e_2 \in A^-(w)$ satisfying $\mathrm{cc}(w, e_1) = \mathrm{cc}(w, e_2)$, it holds that $\mathrm{cc}(v, e_1) = \mathrm{cc}(v, e_2)$, where v is the second endpoint of e_1 and e_2. Then, we say f is *uniquely restrictive* at w. Furthermore, if f is uniquely restrictive at each $w \in V(G)$, then we simply say that f is *uniquely restrictive*.

An informal way of describing unique restrictiveness would be to say that if we color a vertex $w \in V(G)$ with some color, say red, then we only want this choice of red at w to contribute to the exclusion of at most one color possibility at each in-neighbor of w. We note that unique restrictiveness is a rather natural idea, as the conflict functions that represent adapted coloring and proper coloring problems are uniquely restrictive; indeed, in both of these settings, choosing the color red at a vertex v can only contribute to the exclusion of the color red at neighbors of v. Furthermore, DP-coloring problems always give uniquely restrictive conflict functions when represented as single-conflict coloring problems since the conflicts between any two vertices form a matching in $C \times C$.

We will also use the following form of the Lovász Local Lemma.

Theorem 2.2 ([16])**.** *Let $\mathcal{B}$ be a set of bad events. Suppose that each event $B \in \mathcal{B}$ occurs with probability at most p, and suppose further that each event $B \in \mathcal{B}$ is mutually independent with all but at most d other events $B' \in \mathcal{B}$. If $ep(d+1) \leq 1$, then with positive probability, no bad event in $\mathcal{B}$ occurs.*

With these preliminaries in place, we have the following theorem, which gives an upper bound on the number of colors needed for a single-conflict coloring of a d-degenerate graph whose conflict function is uniquely restrictive. Since any conflict function on a simple graph is uniquely restrictive, the following theorem implies Theorem 1.7 and hence gives an affirmative answer to Question 1.5. Our main tool for this theorem will be the application of the Lovász Local Lemma used by Aharoni, Berger, Chudnovsky, Havet, and Jiang [1] in which each vertex receives a random inventory of colors.

Theorem 2.3. *Let G be an oriented graph of maximum degree Δ with a maximum out-degree of at most d. Let C be a set of k colors, and let each arc $e \in A(G)$ have an associated conflict $f(e) \in C^2$. If f is uniquely restrictive, and if $k \geq 2\sqrt{d\left[1 + \log((d+1)\Delta)\right]}$, then G has a single-conflict coloring with respect to f and C.*

Proof. First, we note that since every subgraph of G has an average degree of at most $2d$, G is $(2d)$-degenerate and hence has a single-conflict coloring whenever $k \geq 2d + 1$. Therefore, we may assume in our proof that $k \leq 2d$.

First, for each vertex $v \in V(G)$, we define a color inventory S_v, and for each color $c \in C$, we add c to S_v independently with probability $p = \frac{k}{2d} \leq 1$. Next, we let S'_v be a copy of S_v. (We will need these copies for technical reasons related to the Lovász Local Lemma.) Then, for each vertex $v \in V(G)$, we consider each outgoing arc e of v, and we write $e = (v, w)$. If, for some color $c \in S_v$, we have $f(e) \in \{(c, c') : c' \in S_w\}$, then we delete c from S'_v. In other words, if the color c at v contributes to the forbidden pair $f(v, w) = (c, c')$ of an outgoing arc $(v, w) \in A^+(v)$, and if $c' \in S_w$, then we delete c from S'_v. Then, for each vertex $v \in V(G)$, we let B_v denote the bad event that after this process, S'_v is empty. We observe that if no bad event occurs, then we may arbitrarily color each vertex v with a color from S'_v to obtain a single-conflict coloring of G. Indeed, if some arc (v, w) is colored with a forbidden pair (c, c') where $c \in S'_v$ and $c' \in S'_v$, then it must follow that c was actually deleted from S'_v, a contradiction.

Now, given a vertex $v \in V(G)$, we calculate the probability that the bad event B_v occurs. For a given color $c \in C$, we write b_c for the number of arcs $e \in A^+(v)$ for which $c = \mathrm{cc}(v, e)$. If c does not belong to S'_v, then either c was never added to S_v, or c was added to S_v and then deleted from S'_v. The probability that c was never added to S_v is equal to $1 - p$, and the probability that c was added to S_v and then deleted from S'_v is at most $b_c p^2$. Therefore, the total probability that $c \notin S'_v$ is at most $1 - p + b_c p^2$. Furthermore, since f is uniquely restrictive, the probabilities of any two given colors being absent from S'_v are independent. Therefore, the probability of the bad event B_v is at most

$$\prod_{c \in C} \left(1 - \left(p - b_c p^2\right)\right) < \exp\left(-\sum_{c \in C}\left(p - b_c p^2\right)\right) = \exp\left(-pk + p^2\sum_{c \in C} b_c\right) = \exp\left(-pk + p^2 d\right).$$

Substituting $p = \frac{k}{2d}$, we see that $\Pr(B_v) < \exp\left(-\frac{k^2}{4d}\right)$. Furthermore, as the bad event B_v involves $d + 1$ vertices (namely v and at most d out-neighbors of v), each of maximum degree Δ, B_v is dependent with fewer than $(d + 1)\Delta$ other bad events. Note that since

we use unmodified inventories S_w to determine whether the copy S'_v is empty, we prevent the dependencies of B_v from spreading past the out-neighbors of v. Therefore, using the Lovász Local Lemma (Theorem 2.2), we see that G receives a single-conflict coloring with positive probability as long as $e(d+1)\Delta \exp\left(-\frac{k^2}{4d}\right) \leq 1$. This inequality holds whenever $k \geq 2\sqrt{d[1 + \log((d+1)\Delta)]}$, which completes the proof. $\qquad\square$

Using Theorem 2.3, we can prove Corollary 1.8, which gives an upper bound on the number of colors needed for a cooperative coloring of a family of degenerate graphs. The proof is available in the full version on arXiv [3].

If G does not have parallel edges, then any conflict function $f : E(G) \to C^2$ must be uniquely restrictive. Then, Theorem 2.3 tells us that $\chi_{\leftrightarrow}(G) \leq 2\left\lceil\sqrt{d\left(1 + \log((d+1)\Delta)\right)}\right\rceil$, which gives an affirmative answer to Question 1.5 for simple graphs.

3   General conflicts

Given an oriented graph G with a conflict function f, we define the *restrictiveness* of f at v as the maximum value r_v for which there exists an r_v-tuple of parallel arcs in $A^+(v)$ whose conflicts form a set $\{(c_1, c^*), (c_2, c^*), \ldots, (c_{r_v}, c^*)\}$, where the first entry in each conflict corresponds to v, where $c^* \in C$ is any single color, and where $c_1, \ldots, c_{r_v}$ are all distinct colors. Then, we say that the *restrictiveness* of f is the maximum restrictiveness r_v of f at v, taken over all vertices $v \in V(G)$. The restrictiveness r of a uniquely restrictive conflict function satisfies $r = 1$. If f is a conflict function on a graph G of edge-multiplicity at most μ, then the restrictiveness r of f satisfies $r \leq \mu$.

Theorem 2.3 gives an upper bound on number of colors needed for a single-conflict coloring given a conflict function with restrictiveness $r = 1$. In this section, we will show in the following theorem that we can also find an upper bound on the number of colors needed for a single-conflict coloring given a conflict function whose restrictiveness r is known but may be greater than 1. Since $r \leq \mu$ for any graph G with edge multiplicity at most μ, the following theorem (proven in the full version [3]) also proves Theorem 1.6, giving an upper bound for $\chi_{\leftrightarrow}(G)$ of d-degenerate graphs G with small edge-multiplicity.

Theorem 3.1. *Let G be an oriented graph of maximum degree Δ with a maximum out-degree of at most d. Let C be a set of k colors, and let each arc $e \in A(G)$ have an associated conflict $f(e)$. If the restrictiveness of f is at most r, and if*

$$k \geq \sqrt{d} \cdot 2^{r/2+2}\sqrt{r}\sqrt{1 + \log((d+1)\Delta)},$$

then G has a single-conflict coloring with respect to f and C.

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