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Effective bounds for induced size-Ramsey numbers of cycles

(Extended abstract)

Domagoj Bradač* Nemanja Draganić* Benny Sudakov*

Abstract

The induced size-Ramsey number $\hat{r}^k_{\text{ind}}(H)$ of a graph H is the smallest number of edges a (host) graph G can have such that for any k-coloring of its edges, there exists a monochromatic copy of H which is an induced subgraph of G. In 1995, in their seminal paper, Haxell, Kohayakawa and Łuczak showed that for cycles, these numbers are linear for any constant number of colours, i.e., $\hat{r}^k_{\text{ind}}(C_n) \leq Cn$ for some $C = C(k)$. The constant C comes from the use of the regularity lemma, and has a tower type dependence on k. In this paper we significantly improve these bounds, showing that $\hat{r}^k_{\text{ind}}(C_n) \leq O(k^{102})n$ when n is even, thus obtaining only a polynomial dependence of C on k. We also prove $\hat{r}^k_{\text{ind}}(C_n) \leq e^{O(k \log k)}n$ for odd n, which almost matches the lower bound of $e^{\Omega(k)}n$. Finally, we show that the ordinary (non-induced) size-Ramsey number satisfies $\hat{r}^k(C_n) = e^{O(k)}n$ for odd n. This substantially improves the best previous result of $e^{O(k^2)}n$, and is best possible, up to the implied constant in the exponent. To achieve our results, we present a new host graph construction which, roughly speaking, reduces our task to finding a cycle of approximate given length in a graph with local sparsity.

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1 Introduction

The Ramsey number $r^k(H)$ of a graph is the smallest integer n such that every k-coloring of the edges of K_n contains a monochromatic copy of H. The notion of Ramsey numbers

*Department of Mathematics, ETH, Zürich, Switzerland. Research supported in part by SNSF grant 200021_196965. E-mail: {domagoj.bradac, nemanja.draganic, benjamin.sudakov}@math.ethz.ch.

is one of the most central notions in combinatorics and it has been studied extensively since Ramsey [23] showed their existence for every graph H. Motivated by this definition, we say that a graph G is k-Ramsey for a graph H if any k-coloring of the edges of (the *host graph*) G, contains a monochromatic copy of H, and we write $G \xrightarrow{k} H$. Using this notation, we have that $r^k(H) = \min\{|V(G)| : G \xrightarrow{k} H\}$.

The notion of Ramsey numbers is measuring the minimality of the host graph in terms of the number of vertices. Are there graphs G with significantly fewer edges than the clique on $r^k(H)$ vertices that are k-Ramsey for H? This general question is captured by the notion of size-Ramsey numbers introduced in 1978 by Erdős, Faudree, Rousseau and Schelp [11]. The size-Ramsey number of a graph H is defined as $\hat{r}^k = \min\{E(G)|G \xrightarrow{k} H\}$. In the last few decades, there has been extensive research on this notion, see, e.g., [3].

One of the main goals is to understand which classes of graphs have size-Ramsey numbers which are linear in their number of edges. Beck [2] showed that this is true for paths, which was later extended to all bounded-degree trees by Friedman and Pippenger [14]. It is also known that logarithmic subdivisions of bounded degree graphs have linear size-Ramsey numbers [6], as well as bounded degree graphs with bounded treewidth [18]. Given all of the mentioned results, it might be tempting to assume that all graphs of bounded degree have linear size-Ramsey numbers. In an elegant paper of Rödl and Szemerédi [25], it was shown that this is not the case. Indeed, they showed that there exist n-vertex cubic graphs which have size-Ramsey numbers at least $n \log^c n$, for a small constant $c > 0$. This bound has only very recently been improved to $cne^{c\sqrt{\log n}}$ for some $c > 0$ by Tikhomirov [26]. For more results see [7] and references therein.

A related studied notion is that of induced size-Ramsey numbers. Given a graph H, the induced size-Ramsey number $\hat{r}^k_{ind}(H)$ is the smallest number of edges a graph G can have such that any k-coloring of G contains a monochromatic copy of H which is an induced subgraph of G. The existence of these numbers is an important generalisation of Ramsey's theorem, proved independently by Deuber [4], Erdős, Hajnal, and Pósa [12], and Rödl [24]. Naturally, this concept is much harder to understand for most classes of target graphs H and much less precise bounds are known than for the (non-induced) size-Ramsey number.

Indeed, already for bounded degree trees we know that the size-Ramsey number is linear in their number of vertices, whereas for its induced counterpart we have no good bounds while we have every reason to believe that the answer should also be linear. Further, the best general upper bound on $\hat{r}^2_{ind}(H)$ for n-vertex graphs H is obtained by Conlon, Fox and Sudakov [19], and is of the order $2^{O(n \log n)}$, while Erdős [10] conjectured that $\hat{r}^2_{ind}(H) \leq 2^{cn}$. In comparison, the bound for Ramsey numbers (and hence also for size-Ramsey numbers) is known to be exponential in the number of vertices of the target graph. Further, it is known that the size-Ramsey number of n-vertex graphs with degree bounded by a constant d, is between $ne^{\Omega(\sqrt{\log n})}$ and $O(n^{2-\frac{1}{d}+\varepsilon})$, proven by Tikhomirov [26], and by Kohayakawa, Rödl, Schacht, and Szemerédi [20], respectively. On the other hand, the best upper bound on the induced size-Ramsey number of these graphs, proved by Fox and Sudakov [13] is of the order $n^{O(d \log d)}$, while the best lower bound is still the bound for the (non-induced) Ramsey number of those graphs, which is often the state of the art for such questions.

For paths it is known that $\Omega(k^2)n \leq \hat{r}^k(P_n) \leq O(k^2 \log k)n$ (see [9, 21] for the lower bound and [22, 8] for the upper bound). In the induced case, by a recent result of Draganić, Krivelevich and Glock [5], we have that $\hat{r}^k_{ind}(P_n) \leq O(k^3 \log^4 k)n$. For cycles, the discrepancy between the size-Ramsey and the induced size-Ramsey number is significantly larger. Indeed, by a recent result of Javadi and Miralaei [17], which improved another recent result by Javadi, Khoeini, Omidi and Pokrovskiy [16], we have $\hat{r}^k(C_n) = O(k^{120} \log^2 k)n$ for even n, and $\hat{r}^k(C_n) = O(2^{16k^2+2\log k})n$ for odd n. On the other hand, the only known upper bound on the induced size-Ramsey numbers of cycles was obtained in the seminal paper of Haxell, Kohayakawa and Łuczak [15]. Their proof uses a technically very involved argument relying on the use of the Sparse Regularity lemma and therefore shows that $\hat{r}^k_{ind}(C_n) \leq Cn$ where $C = C(k)$ has a tower type dependence on k.

In this paper, we prove the following theorem which quite significantly improves the tower-type bounds of Haxell, Kohayakawa and Łuczak.

Theorem 1.1. *For any integer $k \geq 1$, there exists $n_0(k)$ such that for all $n \geq n_0(k)$, the following holds.*

a) If n is even, then $\hat{r}^k_{ind}(C_n) = O(k^{102})n$.

b) If n is odd, then $\hat{r}^k_{ind}(C_n) = e^{O(k \log k)}n$.

While the focus of this paper is on induced size-Ramsey numbers of cycles, our method can be also used to substantially improve the upper bound for the non-induced case as well. Our next result gives an essentially tight estimate for the size-Ramsey numbers of odd cycles.

Theorem 1.2. *For any integer $k \geq 1$, there exists $n_0(k)$ such that for all $n \geq n_0(k)$, we have $\hat{r}^k(C_n) = e^{O(k)}n$.*

The best known lower bound for size-Ramsey numbers of even cycles comes from the bound for paths, which is of the order $\Omega(k^2)n$ [9, 22]. In the odd case, there is a simple construction of a coloring which gives a lower bound of $2^{k-1}n$ (see [17]), showing that the second result in Theorem 1.1 is tight up to an $O(\log k)$ factor in the exponent, while the bound in Theorem 1.2 is tight up to a constant factor in the exponent.

We remark that, as in [15], our proofs can easily be adapted to provide monochromatic induced cycles of all (even) lengths between $C \log n$ and n for some constant C depending only on k. We also note that our bound on the size-Ramsey number of even cycles $\hat{r}^k(C_n) \leq \hat{r}^k_{ind}(C_n) = O(k^{102})n$ can be further improved significantly, using the same methods, but we chose not to present that here.

2 Proof outline

The main idea behind our proof is the following: consider a binomial random graph $G \sim \mathcal{G}(N, C/N)$, where $N = C'n$ and C, C' are appropriately chosen large constants. Let G be adversarially k-edge-colored. Then, it is easier to find an induced monochromatic cycle of length in $[0.9n, 1.1n]$, say, then of length precisely n. Our host graph is constructed to take advantage of this.

In the rest of the outline we focus on the proof of the induced odd case (Theorem 1.1 b)) and at the end we outline the changes needed for the other two statements.

Given k, we find a fixed "gadget" graph $F = F(k)$ which is k-induced-Ramsey for a 5-cycle. We denote $s = v(F)$. We construct an s-uniform N-vertex hypergraph H by taking CN random hyperedges. We clean H so it does not have any short Berge cycles so, in particular, it is linear. Then we construct our host graph Γ by placing an isomorphic copy of F inside every hyperedge of H. By definition, inside every copy of F, there is a monochromatic induced copy of C_5. The main object we work with will be an auxiliary k-edge-coloured graph G on the same vertex set as Γ. For each placed copy of F in Γ, in G we put an edge between a single pair of vertices which are at distance 2 in one of the induced monochromatic copies of C_5 in the copy of F, and colour this edge with the colour of that cycle.

Now, suppose we find a monochromatic, say red, cycle Q of length $\ell \in [n/3, n/2]$ in G. By definition, each edge of Q corresponds to an induced 5-cycle in Γ, where the endpoints of the edge are at distance 2 in the cycle. For each of these 5-cycles, we can choose either a path of length 2 or a path of length 3 in G to obtain a red cycle Q' of length exactly n in Γ (see Figure 1). The main technical difficulty is in obtaining certain properties of Q such that the resulting cycle Q' is induced in Γ.

More precisely, the following will be sufficient. Recall that every edge $e \in E(G)$ comes from a hyperedge in H which we denote by $h(e)$. Suppose Q is a cycle in G with edges $e_1, \dots, e_\ell$ such that no hyperedge apart from $h(e_1), \dots, h(e_\ell)$ in H intersects $\bigcup_{i \in [\ell]} h(e_i)$ in more than one vertex. Further, suppose that each $h(e_i)$ only intersects $h(e_{i-1})$ and $h(e_{i+1})$ among the mentioned hyperedges. Then, it is not difficult to see that the cycle Q' obtained as above is induced in Γ. We will call such a cycle Q *good*.

Let us now explain how to find an induced monochromatic cycle of length between $n/2$ and $n/3$ in a k-edge-colored graph $G \sim \mathcal{G}(N, C/N)$ with $N = C'n$ for some large constants C, C'. Our real task is more involved as we require a stronger condition on the found cycle as discussed above, since we are not working with a binomial random graph. However, most of the ideas can be described through the lens of this simpler problem.

We now sketch how to find a monochromatic induced cycle of length between $n/2$ and $n/3$ in $G \sim \mathcal{G}(N, C/N)$. The proof strategy is illustrated in Figure 2. By standard results, it is not difficult to clean G without losing many edges, so that it has no cycles of length $O(1)$. Further, we also know that it is locally sparse, that is, all sets U of size $|U| \leq \varepsilon N$ span at most $\frac{3}{2}|U|$ edges, where $\varepsilon > 0$ is some constant depending on C. We consider the subgraph corresponding to the densest colour class, say red and using a result of Krivelevich [21], we find inside it a large expanding subgraph G'. Draganić, Glock and Krivelevich [5] showed using a modified DFS algorithm that under the given assumptions, G' has a red induced path P of length $2n/5$ and we adapt their argument to our setting. Given such a red induced path of length $2n/5$, from the endpoints we construct two trees

T_1, T_2 each of depth $O(\log N)$ and with $\Omega(\varepsilon N)$ leaves. Moreover, we do it in such a way that any path containing the initial endpoints is good, i.e. if there is a red edge connecting two vertices in different trees, it closes a good cycle in G'. Let $W = V(P) \cup V(T_1) \cup V(T_2)$ and remove from it a large constant number of the last layers in T_1 and T_2, so that the resulting W is small enough compared to the leaf sets of T_1 and T_2. Denote by R_1 and R_2 the vertices in the deleted layers in T_1 and T_2, respectively. Finally, using the expanding properties of G', we may expand from the sets R_1 and R_2, while avoiding vertices which are incident to W until the two balls around R_1 and R_2 of large enough constant diameter intersect, and thus we close a cycle of desired length. Using the girth assumption on our graph it is not difficult to show that this cycle is induced.

Let us now comment on the differences in the proofs for the three different statements. In the odd induced case, we can take F to be Alon's [1] celebrated construction of a dense pseudorandom triangle-free graph on $e^{\Theta(k \log k)}$ vertices. We will prove that, every k-edge-colouring of that graph will contain an induced monochromatic C_5. However, when n is even, we can instead take F to be k-induced-Ramsey for a 6-cycle with only $O(k^6)$ vertices by taking a sufficiently dense bipartite C_4-free graph. Again, in each copy of F, we find a monochromatic induced 6-cycle and connect two vertices at distance 2 on the cycle to form our auxiliary graph. The same argument as above shows that given a monochromatic cycle of length ℓ in the auxiliary graph G, we can find a monochromatic cycle of any even length between 2ℓ and 4ℓ in Γ. Finally, for the odd non-induced case, we can take F to be the complete graph on $2^k + 1$ vertices. It is easy to see that any k-edge coloring of that graph has a monochromatic odd cycle. For simplicity, we take the most common length L among those cycles, and for each of these L-cycles, we connect two vertices at distance $(L-1)/2$ on the cycle to form the auxiliary graph. Then, a monochromatic good cycle of length between $2n/(L-1)$ and $2n/(L+1)$ in the auxiliary graph yields a monochromatic cycle of length n in our host graph. This required extra precision in the length of the good cycle in the auxiliary graph will only cost us a factor of $2^{O(k)}$ in the number of copies of F we use in our construction.

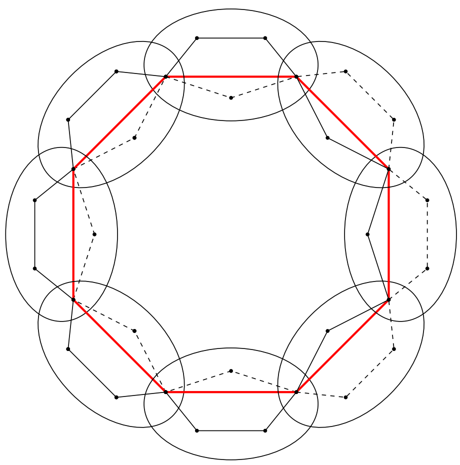

Figure 1: Transforming an 8-cycle in the auxiliary graph (thick red edges) into a 21-cycle in the original graph by using 5 paths of length 3 and 3 paths of length 2.

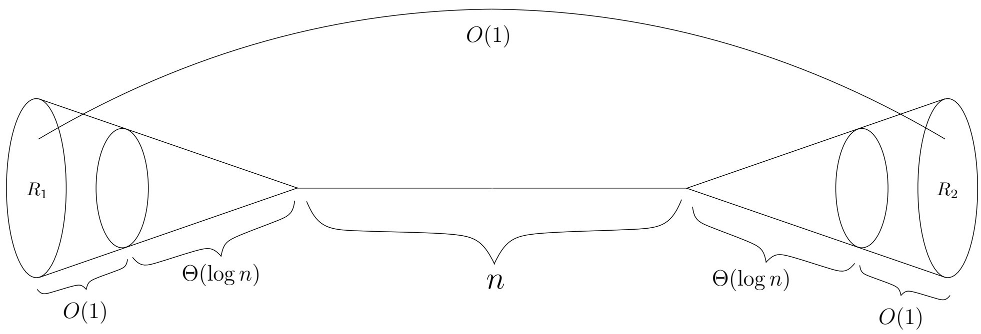

Figure 2: Building an induced cycle

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