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A resolution of the Kohayakawa–Kreuter conjecture for the majority of cases

(Extended abstract)

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Abstract

For graphs $G, H_1, \ldots, H_r$, write $G \to (H_1, \ldots, H_r)$ to denote the property that whenever we r-colour the edges of G, there is a monochromatic copy of H_i in colour i for some $i \in \{1, \ldots, r\}$. Mousset, Nenadov and Samotij proved an upper bound on the threshold function for the property that $G(n,p) \to (H_1, \ldots, H_r)$, thereby resolving the 1-statement of the Kohayakawa–Kreuter conjecture. We extend upon the many partial results for the 0-statement, by resolving it for a large number of cases, which in particular includes (but is not limited to) when $r \geq 3$, when H_2 is strictly 2-balanced and not bipartite, or when H_1 and H_2 have the same 2-densities.

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1 Introduction

Let $r \in \mathbb{N}$ and $G, H_1, \ldots, H_r$ be graphs. We write $G \to (H_1, \ldots, H_r)$ to denote the property that whenever we colour the edges of G with colours from the set $[r] := \{1, \ldots, r\}$ there exists $i \in [r]$ and a copy of H_i in G monochromatic in colour i. Ramsey's theorem states that for any $H_1, \ldots, H_r$, there exists n_0 such that for all $n \geq n_0$, $K_n \to (H_1, \ldots, H_r)$. Since

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the property $G \to (H_1, \ldots, H_r)$ is monotone, a result of Bollobás and Thomason [1] implies that there must exist a threshold function p_0 for the property that the binomial random graph $G(n,p)$ (which has n vertices and contains each possible edge independently with probability p) satisfies $G(n,p) \to (H_1, \ldots, H_r)$. Rödl and Ruciński [11, 12, 13] famously located the threshold for the symmetric case, while Kohayakawa and Kreuter [4] gave a conjecture for the threshold for the asymmetric case.

1.1 Notation

Before we can state these thresholds we require some notation. Let $G = (V, E)$ be a graph. We denote the number of vertices in G by $v_G := |V(G)|$ and the number of edges in G by $e_G := |E(G)|$. Moreover, for graphs H_1 and H_2 we let $v_1 := |V(H_1)|$, $e_1 := |E(H_1)|$, $v_2 := |V(H_2)|$ and $e_2 := |E(H_2)|$.

Let H be a graph. We define

$$d(H) := \begin{cases} e_H/v_H & \text{if } v_H \geq 1, \\ 0 & \text{otherwise;} \end{cases}$$
$$m(H) := \max\{d(J) : J \subseteq H\}.$$

We define the *arboricity* (also known as the 1-density measure) by

$$d_1(H) := \begin{cases} e_H/(v_H - 1) & \text{if } v_H \geq 2, \\ 0 & \text{otherwise;} \end{cases}$$
$$ar(H) = m_1(H) := \max\{d_1(J) : J \subseteq H\}.$$

In [11], Rödl and Ruciński introduced the following so-called *2-density measure*:

$$d_2(H) := \begin{cases} (e_H - 1)/(v_H - 2) & \text{if } H \text{ is non-empty with } v_H \geq 3, \\ 1/2 & \text{if } H \cong K_2, \\ 0 & \text{otherwise;} \end{cases}$$
$$m_2(H) := \max\left\{d_2(J) : J \subseteq H\right\}.$$

We say that H is *strictly 2-balanced* if for all proper subgraphs $J \subset H$, we have $d_2(J) < m_2(H)$.

Regarding asymmetric Ramsey properties, in [4], Kohayakawa and Kreuter introduced the following asymmetric versions of d_2 and m_2. Let H_1 and H_2 be any graphs, and define

$$d_2(H_1, H_2) := \begin{cases} \frac{e_1}{v_1 - 2 + \frac{1}{m_2(H_2)}} & \text{if } H_2 \text{ is non-empty and } v_1 \geq 2, \\ 0 & \text{otherwise;} \end{cases}$$
$$m_2(H_1, H_2) := \max\left\{d_2(J, H_2) : J \subseteq H_1\right\}.$$

We say that H_1 is *strictly balanced w.r.t.* $d_2(\cdot, H_2)$ if for all proper subgraphs $J \subset H_1$ we have $d_2(J, H_2) < m_2(H_1, H_2)$.

The relevance of strictly balanced graphs is as follows. Let H_1, H_2 be graphs with $m_2(H_1) \geq m_2(H_2)$. We call (H_1, H_2) a *heart* if

- H_2 is strictly 2-balanced,
- when $m_2(H_1) = m_2(H_2)$, H_1 is strictly 2-balanced,
- when $m_2(H_1) > m_2(H_2)$, H_1 is strictly balanced w.r.t. $d_2(\cdot, H_2)$.

It is easy to show that for any pair of graphs (H_1, H_2) with $m_2(H_1) \geq m_2(H_2)$, there exists a heart (H_1', H_2') with

- $H_i' \subseteq H_i$ for each $i \in [2]$,
- $m_2(H_2') = m_2(H_2)$,
- $m_2(H_1', H_2') = m_2(H_1, H_2)$ if $m_2(H_1) > m_2(H_2)$, and
- $m_2(H_1') = m_2(H_1)$ if $m_2(H_1) = m_2(H_2)$.

We call this pair a *heart of* (H_1, H_2). Now observe that in order to prove that $G \not\to (H_1, H_2)$, it suffices to prove that $G \not\to (H_1', H_2')$ for some heart (H_1', H_2') of (H_1, H_2), since any colouring avoiding a monochromatic copy of a subgraph of some H clearly avoids a monochromatic copy of H itself.

1.2 Previous and new results

We can now state the aforementioned symmetric random Ramsey theorem and asymmetric random Ramsey conjecture.

Theorem 1.1 (Rödl and Ruciński [11, 12, 13]). *Let $r \geq 2$ and let H be a non-empty graph such that at least one component of H is not a star or, when $r = 2$, a path on 3 edges. Then there exist positive constants $b, B > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}[G_{n,p} \to (\underbrace{H, \ldots, H}_{r \text{ times}})] = \begin{cases} 0 & \text{if } p \leq bn^{-1/m_2(H)}, \\ 1 & \text{if } p \geq Bn^{-1/m_2(H)}. \end{cases}$$

Note that the assumption on the structure of H is necessary, see e.g. [8] for details.

Conjecture 1.2 (Kohayakawa and Kreuter [4]). *Let $r \geq 2$ and suppose that $H_1, \ldots, H_r$ are non-empty graphs such that $m_2(H_1) \geq m_2(H_2) \geq \cdots \geq m_2(H_r)$ and $m_2(H_2) > 1$. Then there exist constants $b, B > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}[G_{n,p} \to (H_1, \ldots, H_r)] = \begin{cases} 0 & \text{if } p \leq bn^{-1/m_2(H_1,H_2)}, \\ 1 & \text{if } p \geq Bn^{-1/m_2(H_1,H_2)}. \end{cases}$$

This statement of the conjecture involves a slight rephrasing of the original statement as per [8], generalising from the case $r = 2$ and including the assumption of Kohayakawa, Schacht and Spöhel [5] that $m_2(H_2) > 1$. This is in order to avoid possible complications arising from H_2 (and/or H_1) being certain forests, such as those excluded in the statement of Theorem 1.1.

The progress on Conjecture 1.2 so far is as follows.

Theorem 1.3. *The* 1*-statement of Conjecture 1.2 holds* ([8])*. Further, the* 0*-statement of Conjecture 1.2 holds for* $(H_1, \ldots, H_r)$ *in each of the following cases. For some heart* (H_1', H_2') *of* (H_1, H_2)*, we have that:*

(i) H_1' *and* H_2' *are both cycles* ([4])*;*

(ii) H_1' *and* H_2' *are both cliques* ([7])*;*

(iii) H_1' *is a clique and* H_2' *is a cycle* ([6])*;*

(iv) H_1' *and* H_2' *are a pair of regular graphs, excluding the cases when (a)* H_1' *is a clique and* H_2' *is a cycle; (b)* H_2' *is a cycle and* $v_1' \geq v_2'$*; (c)* $(H_1', H_2') = (K_3, K_{3,3})$ ([3])*.*

Note that (i)–(iii) above were only stated for (H_1, H_2) of the precise form of (H_1', H_2') stated (i.e. for (i), with H_1 and H_2 themselves both cycles). However, note that each such pair is a heart itself, so the theorem, via the remark at the end of Section 1.1, extends to the cases indicated.

Our main result is that we can vastly extend the number of cases for which the 0-statement holds:

Theorem 1.4. *The* 0*-statement of Conjecture 1.2 holds for the following cases:*

(i) When $r \geq 3$*, i.e. we have at least* 3 *graphs* H_1, H_2, H_3*;*

(ii) When $m_2(H_1) = m_2(H_2)$*;*

(iii) When there exists a heart (H_1', H_2') *of* (H_1, H_2) *such that* $\chi(H_2') \geq 3$ *or* $m(H_2') > 2$ *or* $ar(H_2') > 2$*.*

We remark that Kuperwasser and Samotij announced independently a proof of case (ii) above at Random Structures and Algorithms 2021/2022.

In the rest of this extended abstract, we shall outline the proof strategy of Theorem 1.4.

2 Proof strategy

Suppose that G is a graph with constant size and $m(G) \leq m_2(H_1, H_2)$. It is easy to show that for $p = cn^{-1/m_2(H_1,H_2)}$, with at least constant probability, G will appear as a subgraph of $G_{n,p}$. Therefore, it better be the case that $G \not\to (H_1, H_2)$. It is natural to ask whether this is in fact the only obstruction in proving a 0-statement.

Question 2.1. *Does it suffice to prove that for all G, H_1, H_2 with $m_2(H_1) \geq m_2(H_2) > 1$ and $m(G) \leq m_2(H_1, H_2)$ we have $G \not\rightarrow (H_1, H_2)$, in order to prove the 0-statement of Conjecture 1.2?*

Further, recalling the definition of hearts earlier, observe that we only need to prove such a statement for (H_1, H_2) which are hearts.

In the symmetric setting, the answer to this question is yes. Additionally, in [9], Nenadov et al. showed that this same phenomenon occurs for a number of symmetric Ramsey-style properties. Therefore, naturally, there have been attempts to answer this question in the asymmetric setting. The first result on this question was given by Gugelmann et al. [2], who additionally proved their result extends to the setting of k-uniform hypergraphs.

Theorem 2.2 ([2])**.** *Let (H_1, H_2) be a heart. If*

(i) a certain family of graphs $\mathcal{F}(H_1, H_2)$ is so-called 'asymmetric-balanced',

(ii) for all G such that $m(G) \leq m_2(H_1, H_2)$ then $G \not\rightarrow (H_1, H_2)$,

then the 0-statement holds for any pair of graphs with heart (H_1, H_2).

See [2] for the precise description of property (i). The next major step was made by the third author, who was inspired by the proof techniques used in [7] for proving Theorem 1.3(ii).

Theorem 2.3 ([3])**.** *Let (H_1, H_2) be a heart. If there exists $\varepsilon > 0$ such that*

(i) a certain family of graphs $\hat{\mathcal{A}}(H_1, H_2, \varepsilon)$ is finite,

(ii) for all $G \in \hat{\mathcal{A}}(H_1, H_2, \varepsilon)$ (which in particular satisfy $m(G) \leq m_2(H_1, H_2) + \varepsilon$) we have $G \not\rightarrow (H_1, H_2)$,

then the 0-statement holds for any pair of graphs with heart (H_1, H_2).

By streamlining the approach of the third author, we are able to prove the desired results.

Theorem 2.4. *Let (H_1, H_2) be a heart. There exists a family $\hat{\mathcal{B}}(H_1, H_2) \subseteq \hat{\mathcal{A}}(H_1, H_2, 0)$ such that if*

(i) $\hat{\mathcal{B}}(H_1, H_2)$ is finite,

(ii) for all $G \in \hat{\mathcal{B}}(H_1, H_2)$ we have $G \not\rightarrow (H_1, H_2)$,

then the 0-statement holds for any pair of graphs with heart (H_1, H_2).

Theorem 2.5. *Let (H_1, H_2) be a heart. Then $\hat{\mathcal{B}}(H_1, H_2)$ is finite.*

Combining Theorems 2.4 and 2.5 shows that the answer to Question 2.1 is yes. In the next section we will give a description of the families $\hat{\mathcal{A}}(H_1, H_2, \varepsilon)$ and $\hat{\mathcal{B}}(H_1, H_2)$ in the case of $m_2(H_1) = m_2(H_2)$.

Now it suffices to prove the colouring result contained in Question 2.1. Note that in the symmetric setting, this colouring result holds in all cases and has a short proof (see e.g. Theorem 3.2 in [10]). By appropriately generalising this result, we can prove the following cases of the asymmetric statement.

Lemma 2.6. *For all G, H_1, H_2 with $m_2(H_1) \geq m_2(H_2) > 1$ and $m(G) \leq m_2(H_1, H_2)$, we have $G \not\to (H_1, \dots, H_r)$ if any of the following conditions are satisfied:*

(i) We have $r \geq 3$, i.e. at least 3 graphs H_1, H_2, H_3;

(ii) We have $m_1(H_2) = m_2(H_2)$;

(iii) We have $\chi(H_2) \geq 3$ or $m(H_2) > 2$ or $ar(H_2) > 2$.

Theorem 1.4 immediately follows from Theorems 2.4 and 2.5 combined with Lemma 2.6. Note that the assumption in Theorem 2.5 that (H_1, H_2) is a heart is actually necessary. This leads to the slightly technical nature of the set of graphs given in Theorem 1.4.

2.1 More details on Theorems 2.4 and 2.5

In the case where $m_2(H_1) = m_2(H_2)$, we have $G \in \hat{\mathcal{A}}(H_1, H_2, \varepsilon)$ if G satisfies

- every edge $e = E(R) \cap E(L)$ for some $R \cong H_1$ and $L \cong H_2$, where $L, R \subseteq G$;
- $m(G) \leq m_2(H_1, H_2) + \varepsilon$;
- G is 2-connected.

For $\ell \geq 4$, define $C_\ell^{K_4}$ to be the graph on 3ℓ vertices and 6ℓ edges obtained by taking a cycle C_ℓ and extending each of its edges to a copy of K_4. This graph satisfies that every edge is the intersection of two triangles, $m(C_\ell^{K_4}) = m_2(K_3, K_3) = 2$ and is 2-connected, and therefore the family $\hat{\mathcal{A}}(K_3, K_3, \varepsilon)$ is not finite for any $\varepsilon > 0$.

The key idea is to refine the family $\hat{\mathcal{A}}(H_1, H_2, \varepsilon)$ so that graphs such as $C_\ell^{K_4}$ are excluded. We now describe $\hat{\mathcal{B}}(H_1, H_2)$ in the case where $m_2(H_1) = m_2(H_2)$. Call an edge e *open in G* if $e \neq E(R) \cap E(L)$ for any $R \cong H_1$ and $L \cong H_2$, where $R, L \subseteq G$. Define $\lambda(G) := v_G - e_G/m_2(H_1, H_2)$. We have that $\hat{\mathcal{B}}(H_1, H_2)$ is the collection of all outputs G that can be returned in the running of algorithm Grow-$\hat{\mathcal{B}}$-Alt (see the figure below) which additionally satisfy $m(G) \leq m_2(H_1, H_2)$.

It is not too hard to see that $\hat{\mathcal{B}}(H_1, H_2) \subseteq \hat{\mathcal{A}}(H_1, H_2, 0)$, and further, $C_\ell^{K_4} \notin \hat{\mathcal{B}}(K_3, K_3)$.

The proof of Theorem 2.4 follows from a careful analysis which is very similar to the proof of Theorem 2.3 in [3]. So finally, we summarise how we prove Theorem 2.5.

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procedure Grow-$\hat{\mathcal{B}}$-Alt($H_1, H_2$)
    $G_0 \leftarrow H_1$
    $i \leftarrow 0$
    while $\lambda(G_i) \geq 0$ do
        if $\exists e \in E(G_i)$ s.t. $e$ is open then
            $\{L, R\} \leftarrow$ any pair $\{L, R\}$ s.t. $L \cong H_2$, $R \cong H_1$ and $E(L) \cap E(R) = \{e\}$
            $G_{i+1} \leftarrow G_i \cup L \cup R$
            $i \leftarrow i + 1$
        else
            return $G_i$
            $e \leftarrow$ any edge of $G_i$
            $\{L, R\} \leftarrow$ any pair $\{L, R\}$ s.t. $L \cong H_2$, $R \cong H_1$, $E(L) \cap E(R) = \{e\}$
                and $E(L) \cup E(R) \not\subseteq E(G_i)$
            $G_{i+1} \leftarrow G_i \cup L \cup R$
            $i \leftarrow i + 1$
        end if
    end while
end procedure
```

Figure 1: The implementation of algorithm Grow-$\hat{\mathcal{B}}$-Alt.

Let $\eta(G)$ be the number of open edges in G. First note that if G is an output of the algorithm Grow-$\hat{\mathcal{B}}$-Alt, then it satisfies $\lambda(G) \geq 0$ and $\eta(G) = 0$. Suppose the following is true:

There exist constants $\kappa, x, y > 0$ depending only on H_1 and H_2 such that in each iteration of the algorithm Grow-$\hat{\mathcal{B}}$-Alt, we either have:

(I) $\lambda(G_i) \leq \lambda(G_{i-1}) - \kappa$ and $\eta(G_i) \geq \eta(G_{i-1}) - x$;

(II) $\lambda(G_i) = \lambda(G_{i-1})$ and $\eta(G_i) \geq \eta(G_{i-1}) + y$.

Then, letting T_1^i and T_2^i count the number of iterations of type I and type II, respectively, to construct G_i, we obtain $\eta(G_i) \geq T_2^i \cdot y - T_1^i \cdot x$. Overall this implies that the number of outputs of the algorithm Grow-$\hat{\mathcal{B}}$-Alt is finite, as required. This is the essence of how we prove finiteness of $\hat{\mathcal{B}}$, however our actual approach involves more technical definitions we wish to avoid here.

For the case where $m_2(H_1) > m_2(H_2)$, the algorithm describing the family $\hat{\mathcal{B}}(H_1, H_2)$ is more complicated, but the overall idea is the same.

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