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Biclique immersions in graphs with independence number 2 *

(Extended abstract)

F. Botler[1] A. Jiménez[2] C. N. Lintzmayer[3] A. Pastine[4]
D. A. Quiroz[2] M. Sambinelli[3]

Abstract

The analogue of Hadwiger's Conjecture for the immersion relation states that every graph G contains an immersion of $K_{\chi(G)}$. For graphs with independence number 2, this is equivalent to stating that every such n-vertex graph contains an immersion of $K_{\lceil n/2 \rceil}$. We show that every n-vertex graph with independence number 2 contains every complete bipartite graph on $\lceil n/2 \rceil$ vertices as an immersion.

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[1]Universidade Federal do Rio de Janeiro – Brasil. [2]Universidad de Valparaíso – Chile. [3]Universidade Federal do ABC – Brazil. [4]CONICET & Universidad Nacional de San Luis – Argentina. E-mails: `fbotler@cos.ufrj.br`, `andrea.jimenez@uv.cl`, `carla.negri@ufabc.edu.br`, `agpastine@unsl.edu.ar`, `daniel.quiroz@uv.cl`, `m.sambinelli@ufabc.edu.br`

1 Introduction

A central problem in graph theory is guaranteeing dense substructures in graphs with a given chromatic number. Hadwiger's Conjecture [13] is one of the most important examples of this pursuit, stating that every loopless graph G contains the complete graph $K_{\chi(G)}$ as a minor (where $\chi(G)$ is the chromatic number of G), thus aiming to generalize the Four Color Theorem. This difficult conjecture is known to hold whenever $\chi(G) \leq 6$ [27], and it is open for the remaining values. Thus, a natural approach is to study whether it holds whenever G is restricted to a particular class of graphs. A class of graphs that has received particular attention (and yet remains open) is that of graphs with independence number 2. A recent survey of Seymour [28] emphasizes the importance of this case, which was first remarked by Mader (see [24]). Plummer, Stiebitz, and Toft [24] gave an equivalent formulation of Hadwiger's Conjecture for such graphs: every n-vertex graph with independence number 2 contains a minor of $K_{\lceil n/2 \rceil}$. Before that, in 1982, Duchet and Meyniel [8] had shown a result that implies that every such graph contains a minor of $K_{\lceil n/3 \rceil}$. Despite much work, see e.g. [11, 14, 15, 32], it is still open whether there is a constant $c > 1/3$ such that every graph with independence number 2 contains a minor of $K_{\lceil cn \rceil}$. Given the difficulty to obtain a clique minor on $\lceil n/2 \rceil$ vertices, Norin and Seymour [23] recently turned into finding dense minors on this amount of vertices. They proved that every n-vertex graph with independence number 2 contains a (simple) minor of a graph H on $\lceil n/2 \rceil$ vertices and $0.98688 \cdot \binom{|V(H)|}{2} - o(n^2)$ edges.

The focus of this paper is a conjecture related to Hadwiger's, concerned with finding graph immersions in graphs with a given chromatic number; this type of substructure is defined as follows. To *split off* a pair of adjacent edges uv,vw amounts to deleting those two edges and adding the edge uw. A graph G is said to contain an *immersion* of another graph H if H can be obtained from a subgraph of G by splitting off pairs of edges and deleting isolated vertices. Notice then that if G contains H as a subdivision, it contains H as an immersion (and as a minor). Immersions have received increased attention in recent years, see e.g. [6, 9, 20, 21, 22, 31], particularly since Robertson and Seymour [26] proved that graphs are well-quasi-ordered by the immersion relation. Much of this attention has been centered around the following conjecture of Abu-Khzam and Langston [1], which is the immersion-analog of Hadwiger's Conjecture.

Conjecture 1 (Abu-Khzam and Langston [1]). *Every loopless graph G contains an immersion of $K_{\chi(G)}$.*

The above conjecture holds whenever $\chi(G) \leq 4$ because Hajós' subdivision conjecture holds in this case, actually giving a subdivision of $K_{\chi(G)}$ [7]. The cases where $\chi(G) \in \{5, 6, 7\}$ were verified independently by Lescure and Meyniel [19] and by DeVos, Kawarabayashi, Mohar, and Okamura [5]. In general, a result of Gauthier, Le, and Wollan [12] guarantees that every graph G contains an immersion of a clique on $\lceil \frac{\chi(G)-4}{3.54} \rceil$ vertices. This result improves on theorems due to Dvořák and Yepremyan [10] and DeVos, Dvořák, Fox, McDonald, Mohar, and Scheide [4].

The case of graphs with independence number 2 has also received attention in regard to Conjecture 1. In particular, Vergara [30] showed that, for such graphs, Conjecture 1 is equivalent to the following conjecture.

Conjecture 2 (Vergara [30])**.** *Every n-vertex graph with independence number 2 contains an immersion of $K_{\lceil n/2\rceil}$.*

As evidence for her conjecture, Vergara proved that every n-vertex graph with independence number 2 contains an immersion of $K_{\lceil n/3\rceil}$. This was later improved by Gauthier *et al.* [12], who showed that every such graph contains an immersion of $K_{2\lfloor n/5\rfloor}$. This last result was extended to graphs with arbitrary independence number [3]. Additionally, Vergara's Conjecture has been verified for graphs with small forbidden subgraphs [25]. The main contribution of this paper is the following result, which states that graphs with independence number 2 contain an immersion of every complete bipartite graph on $\lceil n/2\rceil$ vertices.

Theorem 3. *Let G be an n-vertex graph with independence number 2, and $\ell \leq \lceil n/2\rceil - 1$ be a positive integer. Then G contains an immersion of $K_{\ell,\lceil n/2\rceil-\ell}$.*

Using an argument due to Plummer *et al.* [24] we can show that this implies the following.

Corollary 4. *Let G be a graph with independence number 2, and $1 \leq \ell \leq \chi(G) - 1$. Then G contains an immersion of $K_{\ell,\chi(G)-\ell}$.*

This result leads us to make the following conjecture, which holds trivially when $\ell = 1$.

Conjecture 5. *If $1 \leq \ell \leq \chi(G) - 1$, then G contains an immersion of $K_{\ell,\chi(G)-\ell}$.*

We denote by $K_{a,b,c}$ the graph that admits a partition into parts of sizes a, b, and c such that any pair of these parts induces a complete bipartite graph. In addition to Corollary 4, as evidence for Conjecture 5, we can prove the following strengthening of the case $\ell = 2$.

Proposition 6. *If $\chi(G) \geq 3$, then G contains $K_{1,1,\chi(G)-2}$ as an immersion.*

We note that Conjecture 5 has its parallel in the minor order. Woodall [33] and, independently, Seymour (see [18]), proposed the following conjecture: every graph G with $\ell \leq \chi(G) - 1$ contains a minor of $K_{\ell,\chi(G)-\ell}$. In [33], Woodall showed that (the list-coloring strengthening of) his conjecture holds whenever $\ell \leq 2$. Kostochka and Prince [18] showed that the case $\ell = 3$ holds as long as $\chi(G) \geq 6503$. Kostochka [16] proved it for every ℓ as long as $\chi(G)$ is very large in comparison to ℓ, and later [17] improved this so that $\chi(G)$ could be polynomial in ℓ, namely, whenever $\chi(G) > 5(200\ell \log_2(200\ell))^3 + \ell$. In fact, the results in [16, 17, 18] obtain the full join $K^*_{\ell,\chi(G)-\ell}$, which is the graph obtained from the disjoint union of a K_ℓ and an independent set on $\chi(G) - \ell$ vertices by adding all of the possible edges between them. This and the above-cited result of Norin and Seymour leads us to make the following conjecture.

Conjecture 7. *Let G be an n-vertex graph with independence number 2, and $1 \leq \ell \leq \lceil n/2 \rceil - 1$. Then G contains a minor of $K_{\ell,\lceil n/2\rceil-\ell}$.*

Note that the result of Kostochka leaves open the balanced case, thus not implying Conjecture 7. Moreover, it is not hard to build a graph that is denser than the minor obtained by the result of Norin and Seymour, and yet does not contain $K_{\lfloor n/4\rfloor,\lceil n/4\rceil}$: take a complete graph on $\lceil n/2\rceil$ vertices and delete $\lfloor n/4\rfloor + 1$ edges incident to the same vertex. Thus Conjecture 7 is not implied by this result either.

The rest of the paper is organized as follows. In Section 1.1 we give a few definitions and present an interesting lemma that is used to prove Theorem 3 in Section 2. Due to space limitations, we only present a sketch of the proof. We refer the interested reader to [2] for its details.

1.1 Preliminaries and notation

Let G be a graph. For $v \in V(G)$ and $S \subseteq V(G)$, we define $E(v, S) = \{vu \in E(G) : u \in S\}$. If A and B are disjoint sets, we let $K_{A,B}$ be the complete bipartite graph with bipartition (A, B). A substructure (subgraph, immersion, minor, etc.) is a *clique* if it is a complete graph, and is a *biclique* if it is a complete bipartite graph. In a manner that is equivalent to the definition given in the introduction, we say that a graph G *contains an immersion of H* if there exists an injection $f : V(H) \to V(G)$ and a collection of edge-disjoint paths in G, one for each edge of H, such that the path P_{uv} corresponding to the edge uv has endpoints $f(u)$ and $f(v)$.

Finally, in our proof, we make use of the following lemma, which we believe to be interesting by itself, and that we could not find in the literature.

Lemma 8. *Let $j \leq k$ be positive integers, and let $C_1, \ldots, C_j \subseteq [n]$ be sets of size k. Let A be a set of size k disjoint from $[n]$. Then there are disjoint matchings $M_1, \ldots, M_j$ in $K_{A,[n]}$ such that M_i matches A with C_i for every $i \in [j]$.*

2 Outline of the proof of Theorem 3

Indeed, we can consider graphs with independence number at most 2. The proof follows by induction on $n + \ell$. Let G be an n-vertex graph with $\alpha(G) \leq 2$ and let $\ell \leq \lceil n/2\rceil - 1$ be a positive integer. Note that the result is easy when $n \leq 4$, so we can assume $n \geq 5$. Note also that it suffices to prove the statement in the case G is *edge-critical*, i.e., that the removal of any edge of G increases its independence number. Now, if $n \leq 4\ell - 2$, then $\lceil n/2\rceil - \ell \leq 2\ell - 1 - \ell < \ell$. Thus, by induction there is an immersion of $K_{\ell',\lceil n/2\rceil-\ell'}$ in G, where $\ell' = \lceil n/2\rceil - \ell$. But this is the desired immersion because $K_{\lceil n/2\rceil-\ell,\lceil n/2\rceil-\lceil n/2\rceil+\ell}$ is isomorphic to $K_{\lceil n/2\rceil-\ell,\ell}$. Thus, from now on, we assume that $n \geq 4\ell - 1$.

Now, suppose that G contains two non-adjacent vertices, say x and y, with at least $\ell - 1$ common neighbors, and let $G' = G - x - y$. If $\ell \leq \lceil n/2\rceil - 2 = \lceil (n-2)/2\rceil - 1$, the induction hypothesis guarantees that G' contains an immersion of $K_{\ell,\lceil (n-2)/2\rceil-\ell}$, which we call H'. Otherwise, if we have $\ell = \lceil n/2 \rceil - 1$ we let H' be an arbitrary set of ℓ vertices. Let L and B be the parts of H' having size ℓ and $\lceil n/2 \rceil - 1 - \ell$, respectively, and let $R = V(G') \setminus (L \cup B)$. As $\alpha(G) = 2$, every vertex in G' is either adjacent to x or to y in G. This is true in particular for the vertices in L. In what follows, we add either x or y to B, in order to obtain the desired immersion of $K_{\ell, \lceil n/2 \rceil - \ell}$. This is immediate if x or y is adjacent to every vertex in L. Thus we may assume that $|E(y, L)|, |E(x, L)| < \ell$. Now, let L_x (resp. L_y) be the set of vertices in L adjacent to x but not to y (resp. to y but not to x); L_c be the set of vertices in L adjacent to both x and y; and O_c be the set of vertices adjacent to both x and y that are not in L. As x is adjacent to every vertex in $L_x \cup L_c$, it is enough to find (edge-disjoint) paths from x to L_y without using edges of H'. Notice that $|L_y| + |L_c| = |E(y, L)| \leq \ell - 1$ and that, by hypothesis, we have $|L_c| + |O_c| \geq \ell - 1$. Thus $|O_c| \geq |L_y|$. Let $O_c = \{o_1, o_2, \ldots, o_{|O_c|}\}$ and $L_y = \{\ell_1, \ell_2, \ldots, \ell_{|L_y|}\}$. For $1 \leq i \leq |L_y|$, we take the path $x o_i y \ell_i$. These paths are as desired. Therefore, we may assume that $|N(u) \cap N(v)| \leq \ell - 2$ for every pair of non-adjacent vertices u, v.

2.1 Consequences of edge-criticality

Recall that G is edge-critical, meaning that the removal of any edge $uv \in E(G)$ creates an independent set of size 3. Hence, for such an edge there is a vertex w that is not adjacent to both u and v. We formalize this argument in the following claim.

Claim 9. *For any $u, v \in V(G)$, we have $uv \in E(G)$ if and only if $N[u] \cup N[v] \neq V(G)$.*

For the rest of the proof, we fix two non-adjacent vertices x and y, and partition $V(G)$ as follows:

▷ $C = N(x) \cap N(y)$, the set of common neighbors of x and y;

▷ $X = \overline{N[y]}$, the set of non-neighbors of y excluding y, which contains x; and

▷ $Y = \overline{N[x]}$, the set of non-neighbors of x excluding x, which contains y.

We observe that $|C| \leq \ell - 2$, and that each of X and Y induces a complete subgraph of G, otherwise we could find an independent set of size 3. Moreover, the edge-criticality of G yields the following claim.

Claim 10. *For every vertex $a \in C$, we have $X, Y \not\subseteq N(a)$.*

2.2 Key vertex sets

Let $X_C \subseteq X$ (resp. $Y_C \subseteq Y$) be the set containing vertices $v \in X$ (resp. $v \in Y$) for which $C \subset N(v)$, and put $\overline{X}_C = X \setminus X_C$ (resp. $\overline{Y}_C = Y \setminus Y_C$). Now, given a vertex a in C, we denote by X_a (resp. Y_a) the set of vertices in X (resp. in Y) that are adjacent to a, and put $\overline{X}_a = X \setminus X_a$ and $\overline{Y}_a = Y \setminus Y_a$. Notice that if $v \in \overline{X}_a$ and $w \in \overline{Y}_a$, then v and w must be adjacent, as the independence number of G is 2. Thus we get $K_{\overline{X}_a, \overline{Y}_a}$ as a subgraph of G.

Note that $X_C \subseteq X_a$ and $\overline{X}_C \supseteq \overline{X}_a$ (resp. $Y_C \subseteq Y_a$ and $\overline{Y}_C \supseteq \overline{Y}_a$) for every $a \in C$. Indeed, we have $X_C = \bigcap_{a\in C} X_a$ (resp. $Y_C = \bigcap_{a\in C} Y_a$) and $\overline{X}_C = \bigcup_{a\in C} \overline{X}_a$ (resp. $\overline{Y}_C = \bigcup_{a\in C} \overline{Y}_a$). The following claim gives bounds and relations on the sizes of some of these sets. This control is the key to build the desired immersion.

Claim 11.

1. *Given $a \in C$, we have that $|X_C| \leq |X_a| \leq \ell-2$ and $|Y_C| \leq |Y_a| \leq \ell-2$. Furthermore, we have $|\overline{X}_a| \geq \lceil n/2\rceil - |Y| + 3$ and $|\overline{Y}_a| \geq \lceil n/2\rceil - |X| + 3$.*

2. *For every $v \in \overline{X}_C$ (resp. $w \in \overline{Y}_C$), we have $|N(v) \cap \overline{Y}_C| > \lceil n/2\rceil - |X|$ (resp. $|N(w) \cap \overline{X}_C| > \lceil n/2\rceil - |Y|$).*

2.3 Constructing the immersion

The rest of the proof is divided into two cases which depend on the sizes of $\overline{X}_C$ and $\overline{Y}_C$. The sets that form the bipartition of the immersion depend on which case we are dealing with. The construction requires more care in the case one of $\overline{X}_C, \overline{Y}_C$ is large, which we sketch here. Say, without loss of generality, that $|\overline{X}_C| \geq \ell$. For the rest of the proof, we fix $a \in C$. By Claim 11(1), we can choose $Y^* \subset \overline{Y}_a$ with $|Y^*| = \lceil n/2\rceil - |X|$, and since $|\overline{X}_C| \geq \ell$, we can choose $X^* \subset \overline{X}_C \setminus \overline{X}_a$ with $|X^*| = \ell - |\overline{X}_a|$. Using Claim 11(1) again, we can show that

$$|X^*| \leq |Y^*|. \tag{1}$$

Since X, Y, and $\overline{X}_a \cup \overline{Y}_a$ induce cliques, G contains all edges joining (i) vertices in $\overline{X}_a$ to vertices in Y^*; (ii) vertices in $\overline{X}_a$ to vertices in $X_a \setminus X^*$; and (iii) vertices in X^* to vertices in $X_a \setminus X^*$. It remains to find edge-disjoint paths joining vertices in X^* to vertices in Y^*. For these paths, we only use edges that are incident to vertices in Y and not to vertices in $\overline{X}_a$; this assures that they are disjoint from the edges already used. Let $X^* = \{v_1, v_2, \ldots, v_{|X^*|}\}$ and $Y^* = \{y_1, y_2, \ldots, y_{|Y^*|}\}$. The first step is to use Lemma 8 to find paths joining each vertex v_i to all vertices in Y^* allowing edges between vertices of Y^* to be used at most twice. Nevertheless, the intersections are relatively few and with a combination of different techniques, we are able to fix them and obtain the desired immersion.

Claim 12. *For each $i \in \{1, 2, \ldots, |X^*|\}$, there is a subgraph $K(v_i)$ which contains an immersion of K_{v_i,Y^*} and satisfies that:*

i) each path of $K(v_i)$ with an endpoint in v_i has length at most 2;

ii) for each path $v_i z y_j$ in $K(v_i)$ we have $z \in \overline{Y}_C$; and

iii) if $i \neq j$ and $uw \in E(K(v_i)) \cap E(K(v_j))$, then there is no $r \notin \{i, j\}$ such that $uw \in E(K(v_r))$, and one path containing uw ends at u while the other ends at w.

Proof. Note that, since $X^* \subseteq \overline{X}_C$, Claim 11(2) assures that for each $i \in \{1, 2, \ldots, |X^*|\}$, we have $|N(v_i) \cap \overline{Y}_C| > \lceil n/2 \rceil - |X| = |Y^*|$. In order to use Lemma 8, we define, for each such i, a set $N_i \subset N(v_i) \cap \overline{Y}_C$ with $|N_i| = |Y^*|$, and a set of auxiliary vertices $A = \{a_1, a_2, \ldots, a_{|Y^*|}\}$ with $N(a_j) = \overline{Y}_C$. By (1), we can apply Lemma 8 to $N_1, \ldots, N_{|X^*|}$ together with A to obtain disjoint matchings $M_1, M_2, \ldots, M_{|X^*|}$ such that M_i matches A to N_i, for $i \in \{1, \ldots, |X^*|\}$. Let $M_i = \{z_{i,1}a_1, \ldots, z_{i,|Y^*|}a_{|Y^*|}\}$ where $z_{i,j} \in N_i$ for every i, j.

For each $v_i \in X^*$, we obtain $K(v_i)$ by using y_j whenever a_j is used in a matching. In other words, for every $1 \leq j \leq |Y^*|$, if $z_{i,j}a_j \in M_i$, then we use the path $v_i z_{i,j} y_j$. Notice that $z_{i,j}$ could be y_j itself. When that is the case, we use the path $v_i y_j$. Formally, we define

$$P(i,j) = \begin{cases} v_i z_{i,j} y_j & \text{if } y_j \neq z_{i,j} \\ v_i y_j & \text{if } y_j = z_{i,j}\,. \end{cases}$$

Notice that $P(i,j)$ may not be edge-disjoint from $P(i,k)$ if $k \neq j$, but this can only happen if $P(i,j) = v_i y_k y_j$ and $P(i,k) = v_i y_j y_k$. If that is the case, we redefine $P(i,j)$ as $v_i y_j$ and $P(i,k)$ as $v_i y_k$. Thus, after doing all the necessary changes, we can assume that $P(i,j)$ is disjoint of $P(i,k)$ whenever $j \neq k$. Finally we define $K(v_i) = \bigcup_{j=1}^{|Y^*|} P(i,j)$ and since the $P(i,j)$'s are edge disjoint each $K(v_i)$ contains an immersion of K_{v_i,Y^*} and clearly satisfies items *i*) and *ii*).

Furthermore, as $M_1, \ldots, M_{|X^*|}$ are disjoint matchings, if $uw \in E(K(v_i)) \cap E(K(v_j))$ for some pair $i \neq j$, then it must be that $u, w \in Y^*$. Let $u = y_h$ and $w = y_k$. Then either $z_{i,h} = y_k$ or $z_{i,k} = y_h$. Assume, w.l.o.g., that $z_{i,h} = y_k$. This means that $z_{i,h}a_h = y_k a_h \in M_i$. Thus $y_k a_h \notin M_r$ for $r \neq i$. This, in turn, implies that $z_{j,k} = y_h$, which means that $y_h a_k \in M_j$ and $y_h a_k \notin M_r$ for $r \neq j$. This proves *iii*). □

Let $K(v_1), \ldots, K(v_{|X^*|})$ be the subgraphs given by Claim 12. We would like the v_i, Y^*-paths on these subgraphs to be the X^*, Y^*-paths in our immersion. Yet, if $i \neq j$, $K(v_i)$ might not be edge disjoint from $K(v_j)$. Fortunately, by Claim 12 *iii*) the intersections are restricted, which we can show to be sufficient for fixing them.

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