Independent dominating sets in planar triangulations

(Extended abstract)

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Abstract

In 1996, Matheson and Tarjan proved that every planar triangulation on n vertices contains a dominating set of size at most $n/3$, and conjectured that this upper bound can be reduced to $n/4$ when n is sufficiently large. In this paper, we consider the analogous problem for independent dominating sets: What is the minimum ε for which every planar triangulation on n vertices contains an independent dominating set of size at most εn? We prove that $2/7 \leq \varepsilon \leq 3/8$.

1   Introduction

Let S be a set of vertices of a graph G. We say that S is *dominating* if each vertex of G is in S or is a neighbor of some vertex in S; and that S is *independent* if no two vertices in S

are adjacent. In particular, any maximal independent set in G is dominating. We denote by $\gamma(G)$ (resp. $\iota(G)$) the cardinality of a minimum dominating (resp. independent dominating) set of G. Note that $\gamma(G) \leq \iota(G)$. Such parameters are known as the *domination number* and the *independent domination number* of G, respectively, and their calculations are known to be NP-complete problems even on planar bipartite graphs with maximum degree 3 [13, Corollary 3]. Therefore, it is natural to explore such parameters in special classes of graphs, or to look for upper and lower bounds on them.

In this paper, we focus on *planar* graphs, i.e., graphs that can be drawn in the plane so that intersections of edges happen only at their ends. By a *plane* graph we mean a planar graph together with a fixed planar drawing of it. For general terminology on planar graphs we refer to the book of Diestel [3]. In particular, a *planar triangulation* is a plane graph in which each face is bounded by a triangle; and a *triangulated disk* or a *near triangulation* is a plane graph in which each face, except possibly its outer face, is bounded by a triangle. In 1996, Matheson and Tarjan [8] proved that every triangulated disk G on n vertices satisfies $\gamma(G) \leq n/3$, and posed the following conjecture.

Conjecture 1 (Matheson–Tarjan, 1996). *For every sufficiently large planar triangulation G on n vertices, we have $\gamma(G) \leq n/4$.*

Several partial results has been proved for Conjecture 1 [6, 7, 9, 10]. The best known general result is due to Špacapan [12], who proved that $\gamma(G) \leq 17n/53$ for every planar triangulation G on $n \geq 6$ vertices. Related results for maximal outerplanar graphs has been given in [2, 11].

We are interested in the analogous problem for the independent domination number: What is the minimum ε such that $\iota(G) \leq \varepsilon n$ for every planar triangulation G on n vertices? In contrast to the domination number, this parameter has not received so much attention on planar triangulations. It is known that, for any planar graph G on $n \geq 10$ vertices, $\iota(G) < 3n/4$ [5, Theorem 6] and that $\iota(G) \leq n/2$ if G is planar and $\delta(G) \geq 2$ [5, Theorem 8]. For an excellent survey on independent dominating sets, see [4].

Now, note that since every Eulerian planar triangulation has chromatic number 3, they contain three disjoint independent dominating sets. Goddard and Henning [5, Question 1] asked whether such three sets exist in any planar triangulation. In particular, this would imply that $\iota(G) \leq n/3$ for every n-vertex planar triangulation G. We state the later statement as a conjecture.

Conjecture 2. *For every planar triangulation G on n vertices, we have $\iota(G) \leq n/3$.*

Our main contribution is the following theorem.

Theorem 3. *For every planar triangulation G on n vertices, we have $\iota(G) < 3n/8$. Moreover, if $\delta(G) \geq 5$, then $\iota(G) \leq n/3$.*

We also show that the bound $3n/8$ cannot be reduced below $2n/7$, by presenting an infinite family of planar triangulations G for which $\iota(G) \geq 2n/7$ (see Theorem 5). Note

that this improves an observation of Goddard and Henning [5, Figure 6], who presented an infinite family of planar triangulations G for which $\iota(G) \geq 5n/19$.

As a starting example, we prove that $\iota(G) \leq 2n/5$ for every planar triangulation G on n vertices due to a relation between r-dynamic and acyclic colorings as follows. A k-*coloring* of a graph G is a partition of $V(G)$ into k independent sets. Each part in such a partition is called a *color class*. A coloring of G is r-*dynamic* if each vertex v has neighbors in at least $\min\{r, d(v)\}$ color classes, where $d(v)$ denotes the degree of v in G; and a coloring of G is *acyclic* if the union of any two of its color classes induces a forest. We use the following result of Goddard and Henning [5, Lemma 4].

Lemma 4 (Goddard–Henning, 2020). *For every graph G on n vertices with $\delta(G) \geq r$ for which there is an r-dynamic k-coloring, we have $\iota(G) \leq (k - r) n/k$.*

Borodin [1] showed that every planar graph admits an acyclic 5-coloring. Let G be a planar triangulation and χ_a be an acyclic 5-coloring of G. Note that $\delta(G) \geq 3$ and the neighborhood of each vertex contains a cycle; hence, because χ_a is an acyclic coloring, every vertex has neighbors in at least three color classes of χ_a. Therefore, χ_a is 3-dynamic and, by Lemma 4, we have $\iota(G) \leq 2n/5$.

2 An improved upper bound

In this section, we prove the main theorem of this paper, Theorem 3. For that, we introduce a concept and settle some notation. For a vertex v in G, denote by $N(v)$ the set of neighbors of v in G. For a vertex set S of G, denote by $N(S)$ the set of all neighbors of vertices in S (that may also include vertices of S), and by $N[S]$ the closed neighborhood of S, that is, $N[S] = N(S) \cup S$.

Proof of Theorem 3. Let G be a planar triangulation on n vertices. The celebrated Four-Color Theorem assures that there exists a 4-coloring for G [3, Theorem 5.1.1]. Let C_1, C_2, C_3, C_4 be the color classes in such a coloring. Each C_i is an independent set. If some C_i is empty, then each of the other color classes is non-empty and dominating. Therefore, the smallest of the three non-empty color classes is an independent dominating set of size at most $n/3 < 3n/8$.

So suppose each C_i is non-empty. For each i, let U_i be the set of vertices that are not dominated by C_i. Note that U_1, U_2, U_3, and U_4 are pairwise disjoint, as the neighborhood of any vertex is colored with at least two colors, distinct from the color used in the vertex. We start by proving a stronger statement on U_1, U_2, U_3, and U_4, namely that

$$N[U_i] \cap U_j = \emptyset \qquad \text{if } i \neq j. \tag{1}$$

Indeed, by contradiction, say $v \in N(U_1) \cap U_2$. Then $v \in C_3 \cup C_4$. Let u be a neighbor of v in U_1. As $v \in U_2$, we conclude that $u \in N(U_2) \cap U_1$, and hence $u \in C_3 \cup C_4$ also. Because G is a planar triangulation, and u and v are adjacent, u and v have a common

neighbor, say w. Since either $v \in C_3$ and $u \in C_4$, or $v \in C_4$ and $u \in C_3$, vertex w has either color 1 or 2, contradicting the fact that $v \in U_2$ and $u \in U_1$.

Let S_i be an independent dominating set of $G[U_i]$ and let $S = S_1 \cup S_2 \cup S_3 \cup S_4$. Because each S_i is a subset of U_i, by (1), the set S is independent in G. Let $H = G - S$, and consider the plane embedding of H induced by G. Because G is a planar triangulation, S is independent, and all neighbors of S are in H, each vertex of S lies in a different face of H. Moreover, the boundary of each face of H is a cycle, hence H is connected. Note that, for each $v \in U_i$, the set $N(v)$ is colored with two colors. Thus, the boundary of each face of H in which a vertex from S lies has an even number of vertices.

Let G' be a connected plane graph on n' vertices and m' edges. For $i \geq 3$, let f_i be the number of faces in G' with i edges on their boundary. Then, $2m' = \sum_{i \geq 3} i \, f_i$ and the number of faces in G' is $f' = \sum_{i \geq 3} f_i$. By Euler's formula [3, Theorem 4.2.9], we have that $n' - m' + f' = 2$, which implies that $n' - (\sum_{i \geq 3} i \, f_i)/2 + \sum_{i \geq 3} f_i = n' - \sum_{i \geq 3} \frac{(i-2)}{2} f_i = 2$. Hence, $f_4 + 2 \sum_{i \geq 6} f_i \leq \sum_{i \geq 4} \frac{(i-2)}{2} f_i \leq \sum_{i \geq 3} \frac{(i-2)}{2} f_i = n' - 2$. So, if $G' = H$, then there are at most $f_4 + \sum_{i \geq 6} f_i$ vertices in S, because vertices of S lie in faces with an even number of vertices on their boundary. Thus, $2|S| \leq 2(f_4 + \sum_{i \geq 6} f_i) \leq n' - 2 + f_4$. Moreover, $n' + |S| = n$ because $H = G - S$. Joining the last two inequalities, we conclude that $3|S| \leq n - 2 + f_4$. Hence, each set $C_i \cup S_i$ is an independent dominating set and

$$\sum_{i=1}^{4} (|C_i| + |S_i|) = n + |S| \leq \frac{4n - 2 + f_4}{3}.$$

So the smallest of these four independent dominating sets has size less than $n/3 + f_4/12$. The number of faces in G is $2n - 4$, and there are $4f_4$ faces of G incident to the vertices of degree 4 in S. Therefore $f_4 \leq (2n-4)/4 < n/2$, and $\iota(G) < n/3 + n/24 = 3n/8$. Moreover, if $\delta(G) \geq 5$, then $f_4 = 0$ and $\iota(G) \leq n/3$. $\square$

3   A lower bound

As far as we know, Theorem 3 might not be tight: we do not know a family of planar triangulations G on n vertices with $\iota(G)$ approaching $3n/8$. We improve the previous lower bound on ε given by Goddard and Henning [5, Figure 6] in the next result.

Theorem 5. *There is an infinite family $\mathcal{F}$ of planar triangulations such that $\iota(G) = 2n/7$ for every $G \in \mathcal{F}$, where n is the number of vertices in G.*

Proof. Consider the diamond graph depicted in Figure 1(a). Let us describe a family $\mathcal{F}$ of planar triangulations using this graph. Each planar triangulation in $\mathcal{F}$ consists of a circular chain of such diamond graphs, as depicted in Figure 1(b), with edges added to result in a planar triangulation. The planar triangulation G_k obtained in this way with k diamond graphs has $n = 7k$ vertices. The squared vertices in Figure 1(b) show an independent dominating set with $2k = 2n/7$ vertices. Note that any independent dominating set in such a planar triangulation G_k must contain at least two vertices in each diamond graph, therefore $\iota(G_k) = 2k = 2n/7$. $\square$
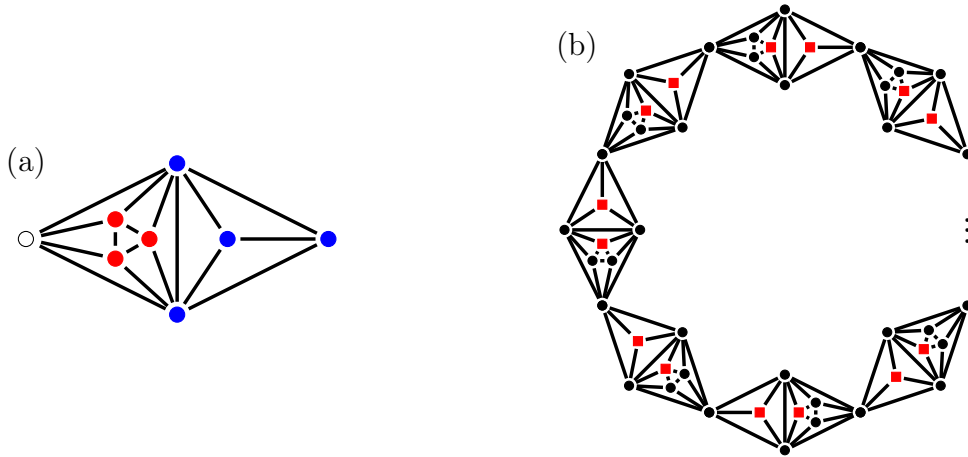
Figure 1: (a) A gadget consisting of seven vertices: the white vertex is not part of the gadget. Any independent dominating set has one of the red vertices, otherwise, being independent, it cannot dominate the three red vertices. Analogously, any independent dominating set has one of the blue vertices, otherwise it does not dominate the middle blue vertex. (b) A graph consisting of a circular chain of gadgets. In each copy of the gadget, two of its vertices are needed in any independent dominating set. The red squared vertices form an independent set with exactly two vertices in each gadget.

4   Further results and concluding remarks

In this section, we explore a few families of planar triangulations for which we can obtain better bounds on their independent domination number. A planar 3-tree is a planar triangulation that can be obtained from a triangle by repeatedly choosing one of its faces and adding a new vertex inside of it while joining this new vertex to the three vertices of the face. It is not hard to prove that any planar 3-tree admits a 4-coloring in which each of its color classes is dominating. Thus $\iota(G) \leq n/4$ for every planar 3-tree on n vertices.

As we observed before, Conjecture 2 is valid for any Eulerian planar triangulation. We can prove a better bound for a particular class of Eulerian triangulations, which we call *recursive Eulerian triangulations*, and define as follows. A recursive Eulerian triangulation is either a triangle, or a triangulation obtained from a recursive Eulerian triangulation by selecting a face and drawing a triangle inside of it while joining each vertex of the selected triangle to the ends of a different edge of the new triangle.

Theorem 6. *For every recursive Eulerian triangulation G on $n \geq 9$ vertices, $\iota(G) < \frac{13n}{42}$.*

Finally, if every vertex of a planar triangulation G on n vertices has odd degree, then every color class of a 4-coloring of G is dominating, hence $\iota(G) \leq n/4$. We can extend this result and show that if G has at least αn odd-degree vertices, then $\iota(G) \leq (2 - \alpha)n/4$, which improves the bound in Theorem 3 when $\alpha \geq 2/7$. Also, Conjecture 2 holds for any n-vertex planar triangulation with at least $2n/3$ odd-degree vertices.

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