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Twin-width and permutations

(Extended abstract)

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Abstract

Inspired by a width invariant on permutations defined by Guillemot and Marx, Bonnet, Kim, Thomassé, and Watrigant introduced the twin-width of graphs, which is a parameter describing its structural complexity. This invariant has been further extended to binary structures, in several (basically equivalent) ways. We prove that a class of binary relational structures (that is: edge-colored partially directed graphs) has bounded twin-width if and only if it is a first-order transduction of a proper permutation class. As a by-product, we show that every class with bounded twin-width contains at most $2^{O(n)}$ pairwise non-isomorphic n-vertex graphs.

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1 Introduction

In this paper we consider the graph parameter *twin-width*, defined by Bonnet, Kim, Thomassé and Watrigant [6] as a generalization of an invariant for classes of permutations defined by Guillemot and Marx [9]. Twin-width was recently studied intensively in the context of many structural and algorithmic questions, such as FPT model checking [6], graph enumeration [3], graph coloring [4], and structural properties of matrices and ordered graphs [5].

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Many well-studied classes of graphs have bounded twin-width: planar graphs, and more generally, any class of graphs excluding a fixed minor, cographs, and more generally, any class of bounded clique-width, etc.

The twin-width of graphs was originally defined using a sequence of 'near-twin' vertex contractions or identifications. Roughly speaking, twin-width measures the accumulated error (recorded via the so-called 'red edges') made by the identifications. To help the reader start forming intuitions, we give a concise definition of the twin-width of a graph.

A *trigraph* is a graph with some edges colored red (while the rest of them are black). A *contraction* (or identification) consists of merging two (non-necessarily adjacent) vertices, say, u, v into a vertex w that is adjacent to a vertex z via a black edge if uz and vz were black edges, or otherwise, via a red edge if at least one of u and v were adjacent to z. The rest of the trigraph does not change. A *contraction sequence* of an n-vertex graph G is a sequence of trigraphs $G = G_n, \ldots, G_1$ such that G_i is obtained from G_{i+1} by performing one contraction (observe that G_1 is the 1-vertex graph). A *d-sequence* is a contraction sequence where all the trigraphs have red degree at most d. The *twin-width* of G is then the minimum integer d such that G admits a d-sequence. See Figure 1 for an example of a graph admitting a 2-sequence.

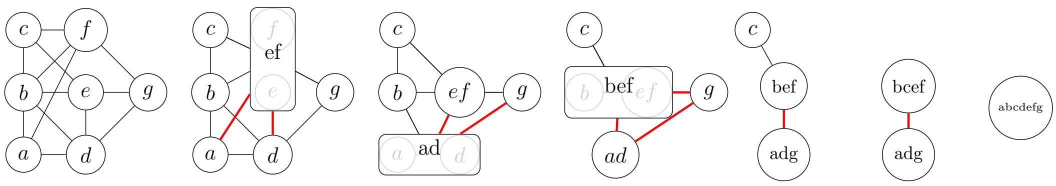

Figure 1: A 2-sequence witnessing that the initial graph has twin-width at most 2.

Our main result deals with relational structures in a way which is consistent (for boundedness) with other definitions used in the literature (see [6], for example).

We show that twin-width can be concisely expressed by special structures, which we call *twin-models*. Twin-models are rooted trees augmented by a set of transversal edges that satisfies two simple properties: minimality and consistency. These properties imply that every twin-model admits a *ranking*, from which we can compute a *width*. The twin-width of a structure then coincides with the optimal width of a ranked twin-model of the structure. While this connection is technical, twin-models provide a simple way to handle classes of binary structures with bounded twin-width. Note that an informal precursor of ranked twin-models appears in [4] in the form of the so-called *ordered union trees* and the realization that the edge set of graphs of twin-width at most d can be partitioned into $O_d(n)$ bicliques where both sides of each biclique are a discrete interval along a unique fixed vertex ordering. The main novelty in the (ranked) twin-models lies in the axiomatization of *legal* sets of transversal edges, which is indispensable to their logical treatment.

This paper is a combination of model-theoretic tools (relational structures, interpretations, transductions), structural graph theory and theory of permutations. Here, by a permutation, we mean a relational structure consisting of two linear orders on the same set (see [1] for a discussion on representations of permutations). Note that this type of representation is particularly adapted to the study of patterns in permutations. The following is the main result of this paper:

Theorem. *A class of binary relational structures has bounded twin-width if and only if it is a first-order transduction of a proper permutation class.*

We recall that a proper permutation class is a set of permutations closed under subpermutations that excludes at least one permutation. Transductions provide a model theoretical tool to encode relational structures (or classes of relational structures) inside other (classes of) relational structures.

The fact that any class of graphs with bounded twin-width is just a transduction of a very simple class (a proper permutation class) is surprising at first glance, and it nicely complements another model theoretic characterization of classes of bounded twin-width: a class of graphs has bounded twin-width if and only if it is the reduct of a dependent class of ordered graphs [5]. It can also be thought of as scaling up the fact that classes of bounded rank-width coincide with transductions of tree orders, and classes of bounded linear rank-width, with transductions of linear orders [8]. On the other hand, twin-models are interesting objects per se and in a way present one of the most permissive forms of width parameters related to trees. Note that for other classes of sparse structures (such as nowhere dense classes or classes with bounded expansion) we do not have such concrete models.

The main result implies that every relational structure on n elements from a class with bounded twin-width can be encoded in a permutation on at most kn elements for some number k. It is then a consequence of [10] that every class of relational structures with bounded twin-width contains at most c^n non-isomorphic structures with n vertices, hence is small (i.e., contains at most $c^n\, n!$ labeled structures with n elements). This extends the main result of [3] while not using the "versatile twin-width" machinery (but only the preservation of bounded twin-width by transductions proved in [6]). This also extends a similar property for proper minor-closed classes of graphs, which can be derived from the boundedness of book thickness, as noticed by McDiarmid (see the concluding remarks of [2]).

The proof of our main result is surprisingly complex and proceeds in several steps, which perhaps add new aspects to the rich spectrum of structures related to twin-width. The basic steps can be outlined as follows (the relevant terminology is formally introduced in the full version of the paper [7]).

We start with a class $\mathscr{C}_0$ of binary relational structures with bounded twin-width. We derive a class $\mathscr{T}$ of twin-models (tree-like representations of the structures using rooted binary trees and transversal binary relations). Replacing the rooted binary trees of the twin-models by binary tree orders, we get a class $\mathscr{F}$ of so-called full twin-models, which we prove has bounded twin-width. This class can be used to retrieve $\mathscr{C}_0$ as a transduction, that is by means of a logical encoding. Using a transduction pairing (generalizing the notion of a bijective encoding) between binary tree orders $\mathscr{O}$ and rooted binary trees ordered by a preorder $\mathscr{Y}^<$ we derive a transduction pairing of the class of full twin-models $\mathscr{F}$ with a class $\mathscr{T}^<$ of ordered twin-models. From the property that the class $\mathscr{G}$ of the Gaifman graphs of the twin-models in $\mathscr{T}$ is degenerate (and has bounded twin-width), we prove a transduction pairing of $\mathscr{T}$ and $\mathscr{G}$, from which we derive a transduction pairing of $\mathscr{T}^<$ and the class $\mathscr{G}^<$ of ordered Gaifman graphs of the ordered twin-models. As a composition of a transduction pairing of $\mathscr{G}^<$ with a class $\mathscr{E}^<$ of ordered binary structures, in which each binary relation induces a pseudoforest and a transduction pairing of $\mathscr{E}^<$ with a class $\mathscr{P}$ of permutations we define a transduction pairing of $\mathscr{G}^<$ and $\mathscr{P}$. As $\mathscr{G}^<$ has bounded twin-width (as it is a transduction of a class with bounded twin-width) we infer that $\mathscr{P}$ avoids a least one pattern. Following the backward transductions, we eventually deduce that $\mathscr{C}_0$ is a transduction of the hereditary closure $\overline{\mathscr{P}}$ of $\mathscr{P}$, which is a proper permutation class.

This proof may be schematically outlined by Figure 2.

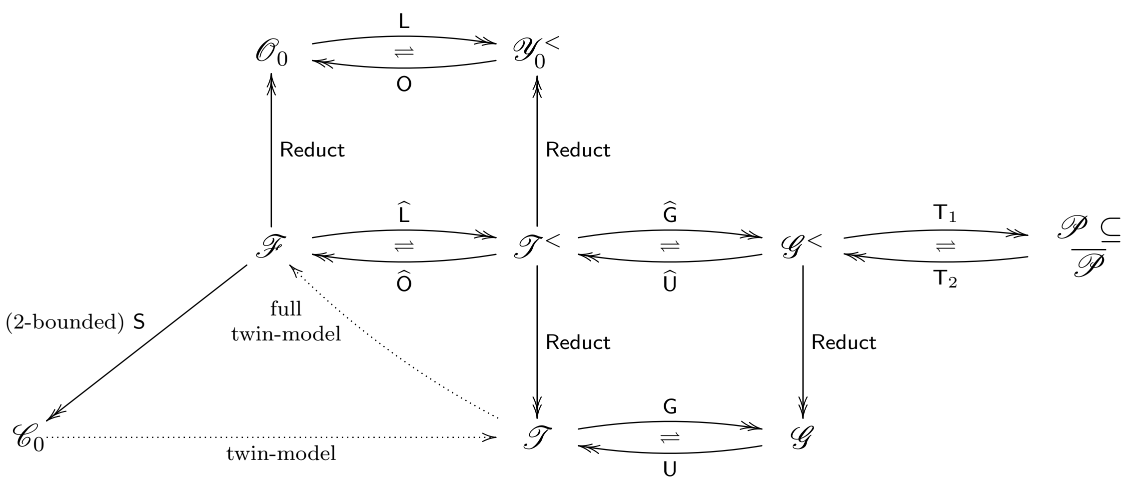

Figure 2: Relations between the classes of structures involved in the proof of the main result.

The full transformation of a graph G into a permutation σ and the inverse transformation (obtained as a transduction) are displayed on Figure 3 on an example.

The full version of this paper is available on arXiv [7].

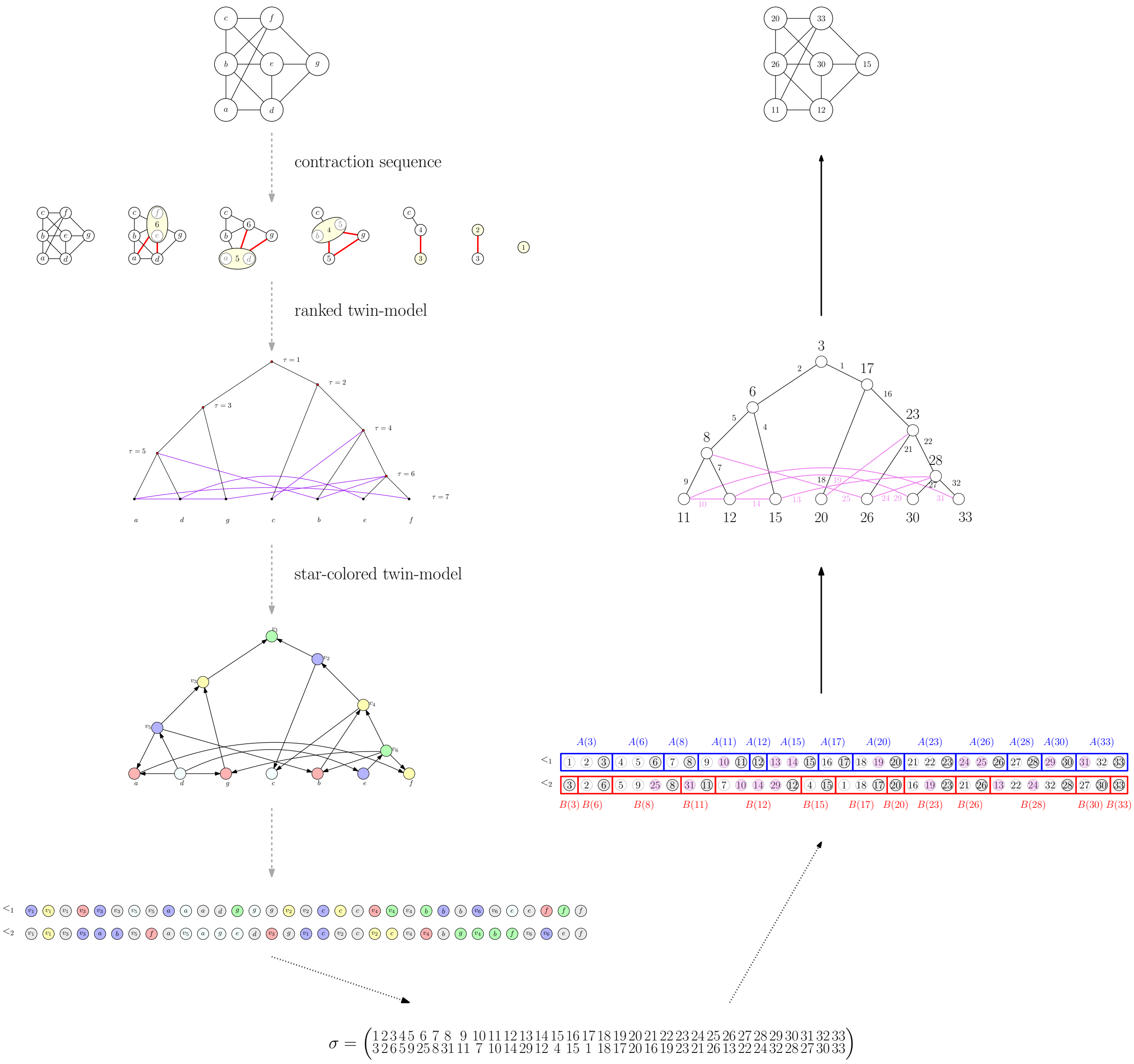

Figure 3: From a graph G to a permutation σ, and back.

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