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The Localization game on locally finite trees

(Extended abstract)

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Abstract

We study the Localization game on locally finite graphs and trees, where each vertex has finite degree. As in finite graphs, we prove that any locally finite graph contains a subdivision where one cop can capture the robber. In contrast to the finite case, for n a positive integer, we construct a locally finite tree with localization number n for any choice of n. Such trees contain uncountably many ends, and we show this is necessary by proving that graphs with countably many ends have localization number at most 2. We finish with questions on characterizing the localization number of locally finite trees.

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1 Introduction

Pursuit-evasion games are most commonly studied on finite graphs, but various studies have also considered the infinite case, such as [3, 12, 14, 15]; see also Chapter 7 of [6]. In this extended abstract, we present the first study of the Localization game on locally finite

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graphs, where the degree of each vertex is finite. Unlike in the finite case, the robber may avoid capture in locally finite graphs by moving along an infinite path.

The Localization game was first introduced for one cop by Seager [16, 17], and was subsequently studied in several papers such as [1, 4, 5, 7, 8, 9]. The game consists of two players playing on a graph. One player controls a set of k *cops* and the other controls a single *robber*. The players play over a sequence of discrete time-steps; a *round* of the game is a move by the cops and the subsequent move by the robber. The players move on alternate time-steps, with the robber going first. The robber occupies a vertex of the graph, and when the robber is ready to move during a round, they may move to a neighboring vertex or remain on their current vertex. The cops' move is a placement of cops on a set of vertices. Note that the cops are not limited to moving to neighboring vertices. In each round, the cops occupy a set of vertices $u_1, u_2, \ldots, u_k$, and each cop sends out a *cop probe*. Each cop probe returns the distance from u_i to the robber. The cops win if they have a strategy to determine, after a finite number of rounds, the location of the robber, at which time we say that the cops *capture* the robber. We assume the robber is *omniscient*, in the sense that they know the entire strategy for the cops. The robber wins by evading capture indefinitely. For a graph G, the *localization number* of G, written $\zeta(G)$, is the smallest cardinal for which k cops have a winning strategy. As locally finite graphs are countable, $\zeta(G)$ is either a positive integer or the first infinite cardinal, $\aleph_0$.

We present new results on the localization number of locally finite (that is, every vertex has finitely many neighbors) graphs and trees, paying particular attention to whether results persist or change from the finite case. As in finite graphs, we prove that any locally finite graph contains a subdivision where one cop can capture the robber. In contrast to the finite case, we construct a locally finite tree with localization number n for any choice of n, where n is a positive integer or $\aleph_0$. These constructions contain uncountably many "infinite branches" or ends. We show that, as in the finite case, trees with countably many ends have localization number of at most two. We close with open questions about characterizing the localization number of locally finite trees.

All graphs considered are simple, connected, and locally finite. The reader is directed to [2, 11] for additional background on graph theory and infinite graphs.

2 Results

Although determining the localization number for general graphs is **NP**-hard [7], the following theorem of Seager characterizes the Localization game on finite trees. Let T_3 be the tree depicted in Figure 1.

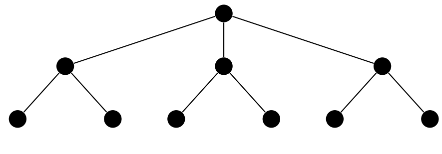

Figure 1: The graph T_3.

Theorem 1 ([17])**.** *If T is a finite tree, then $\zeta(T) = 1$ if and only if T is T_3-free, and otherwise, $\zeta(T) = 2$.*

The Localization game on locally finite trees has received far less attention. While proving a result for finite graphs, Haselgrave, Johnson, and Koch gave the first theorem extending the Localization game to an infinite tree.

Theorem 2 ([13])**.** *The infinite Δ-regular tree T_Δ satisfies $\zeta(T_\Delta) \geq \lfloor \frac{\Delta^2}{4} \rfloor$.*

As a consequence of Theorems 1 and 2, we note that locally finite trees offer a richer spectrum of localization numbers than finite trees. In our first contribution, we show that for any choice of n, including $\aleph_0$, there is a locally finite tree with localization number n.

Theorem 3. *If n is a positive integer or $n = \aleph_0$, then there is a locally finite tree T with $\zeta(T) = n$.*

While the full proof of Theorem 3 will be given in the full paper, we sketch it here. Fix $n > 2$ an integer. For locally finite trees T with $\zeta(T) = n$, consider the subdivision of the infinite $n(n-1)$-regular tree, where each edge is subdivided $n-1$ times. A set of n cops can spend $n-1$ rounds determining which subtree contains the robber. This allows them to move n vertices towards the robber, who can only move $n-1$ away, so the cops eventually overtake and capture the robber. The robber can evade $n-1$ cops by playing on an unprobed branch for at least n rounds. This guarantees the robber can choose any fixed distance d to stay from every probe, avoiding capture. For the case of $n = \aleph_0$, note that Theorem 2 allows us to construct a graph on which the robber can evade any finite number of cops.

Theorem 2 tells us the infinite $n(n-1)$-regular tree requires $\Omega(n^4)$ cops, but subdividing reduced the number of required cops to n. This technique was studied in finite graphs, where it is known that every finite graph G has a subdivision G' such that $\zeta(G') = 1$; see [10]. An analogous result holds for locally finite graphs.

Theorem 4. *For every locally finite graph G, there is a subdivision G' of G such that $\zeta(G') = 1$.*

Unlike in the approach given in [10] in the finite case, we subdivide different edges a different number of times. We defer the complete proof to the full paper.

Locally finite trees have infinite paths where the robber may evade capture. Bearing this in mind, we use the setting of ends to formalize our approach to the localization number of locally finite trees. A *ray* is an infinite one-way path. An *end* is an equivalence class of rays with the property that for any finite set of vertices S, each equivalent ray is in the same component of $G - S$. Different ends can be separated by removing finitely many vertices, matching our concept of separate infinite branches. The theory of ends in general locally finite graphs is complex (see [11]); in the context of locally finite trees, however, we may view an end as an infinite branch of the tree with finite subtrees attached to each vertex.

The locally finite trees considered so far all contain the infinite binary tree as a minor, and such graphs have uncountably many ends. Containing the infinite binary tree as a minor is equivalent to having uncountably-many ends [11].

In the full paper, we will prove the following.

Theorem 5. *If T is a locally finite tree with finitely many ends, then $\zeta(T) \leq 2$.*

Perhaps surprisingly, two cops still have a winning strategy in case there are countably many ends.

Theorem 6. *If T is a locally finite tree with countably many ends, then $\zeta(T) \leq 2$.*

For the proof of Theorem 6, we use transfinite induction on a certain ordinal labeling of ends. The base case for the induction uses Theorem 5.

Proof. Given a locally finite rooted tree T and the corresponding tree order where for $u, v \in V(T)$, $u < v$ if and only if u is on the unique path from v to the root, a *recursive pruning* is a labeling of the vertices of T by ordinals where the collection of vertices that receive ordinal α are those which, after removing all vertices with label $\beta < \alpha$, have up-closures that form chains. In other words, after pruning all vertices labeled so far, assign label α to all vertices after the point where any path or ray starting at the root stops branching. We let T_α be the tree resulting from pruning all vertices with label $\beta < \alpha$, and note that the process of recursive pruning ensures T_α is connected for all α.

For more background on recursive prunings, we direct the reader to [11, Chapter 8]. Trees have a recursive pruning if and only if they do not contain a subdivision of the infinite binary tree [11, Proposition 8.5.1]; such trees are the only examples of "infinite branching" where some vertices cannot be labeled. Thus, every rooted tree with countably many ends has a recursive pruning.

Consider an ordinal labeling of the vertices of T by a recursive pruning. Each end of T contains a ray such that the labels of vertices along that ray are weakly decreasing; otherwise, there would be some α for which T_α is not connected. Since decreasing sets of ordinals are finite, each such ray only contains vertices with finitely many labels, and thus, among those labels that occur infinitely often, one must be largest. If ϵ is an end of T, then we call the largest label that occurs infinitely often on the corresponding ray the *end label* of ϵ.

We claim that if an ordinal α is the supremum of the end labels among the (possibly countably many) ends of T, then there are finitely many ends with end label α. After pruning vertices that received label α, the resulting tree $T_{\alpha+1}$ is connected. If $T_{\alpha+1}$ is empty, then T_α contained no vertices of degree greater than two and thus contained at most two ends, each of which had end label α. If $T_{\alpha+1}$ contains finitely many vertices, then as T is locally finite, each vertex was adjacent to finitely many rays where each vertex received label α, and so there are finitely many such ends. Finally, if $T_{\alpha+1}$ contains infinitely many vertices, it must contain an end [11, Proposition 8.2.1], and the vertices on that end must receive a label larger than α, so α was not the supremum of the end labels.

We next show that the supremum α of the end labels is in fact a maximum. Let $\alpha_1, \alpha_2, \ldots$ infinite be a strictly increasing sequence of ordinals such that there are ends $\epsilon_1, \epsilon_2, \ldots$ where ϵ_i has end label α_i. We will show there is at least one end with end label β such that for all i, $\beta > \alpha_i$; therefore, every infinite chain of increasing end labels has a maximal element. By Zorn's lemma, T has an end with maximum end label.

To find an end with end label β, first note that for every end ϵ_i, we can find an infinite ray r_i beginning at some arbitrarily chosen root of T, say v_0, which belongs to the end ϵ_i. As T is locally finite, v_0 has finitely many neighbors. One of these neighbors say v_1, must be contained in r_i for infinitely many i. We repeat this compactness-type argument to find a ray $v_0 v_1 \cdots$ such that each vertex is contained in infinitely many of the r_i. Each v_i must, therefore, receive a label larger than each α_i. Hence, the end containing this ray must have end label $\beta > \alpha_i$ for each i.

We prove that two cops have a winning strategy by transfinite induction on the largest end label in the recursive pruning of T. For the base case, if T contains no ends, or if the largest end label is 1, then given that there are finitely many ends with end label 1, the result follows from Theorem 5.

Assume now that the theorem holds if the largest end label is strictly less than α and let T be a tree with the largest end label α. By implementing a strategy similar to that used to prove Theorem 5, two cops repeatedly restrict the robber's access to ends with label α until the robber is trapped on a subgraph with ends which have label less than α. At this point, the cops have a winning strategy by the induction hypothesis, and the theorem follows. $\square$

3 Further Directions

Given Theorem 6, it is natural to ask if there is a version of Theorem 1 for locally finite trees with countably many ends. Unlike in the finite case, T_3 is not the only obstruction to a locally finite tree having localization number one. One example is the *doubly infinite comb* graph T_1^∞ consisting of a double ray with a leaf attached to each vertex; see Figure 2.

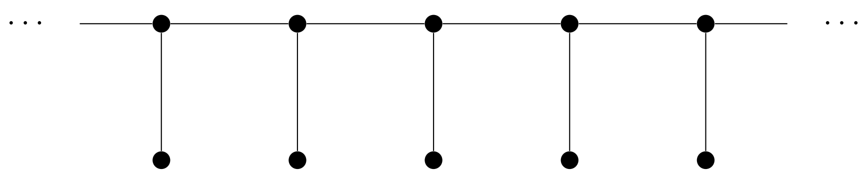

Figure 2: The graph T_1^∞.

The tree T_1^∞ is locally finite, T_3-free tree satisfying $\zeta(T_1^\infty) = 2$. We may show that the tree T_1^∞ is *minimal* in the sense that deleting any edge results in a tree T with $\zeta(T) = 1$. An interesting problem is determining the minimal locally finite trees with countable (or even finite) ends and localization number 2.

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