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On a recolouring version of Hadwiger's conjecture

(Extended abstract)

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Abstract

We prove that for any $\varepsilon > 0$, for any large enough t, there is a graph G that admits no K_t-minor but admits a $(\frac{3}{2} - \varepsilon)t$-colouring that is "frozen" with respect to Kempe changes, i.e. any two colour classes induce a connected component. This disproves three conjectures of Las Vergnas and Meyniel from 1981.

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1 Introduction

In an attempt to prove the Four Colour Theorem in 1879, Kempe [7] introduced an elementary operation on the colourings[1] of a graph that became known as a Kempe change. Given a k-colouring α of a graph G, a *Kempe chain* is a maximal bichromatic component[2]. A *Kempe change* in α corresponds to swapping the two colours of a Kempe chain so as to obtain another k-colouring. Two k-colourings are *Kempe equivalent* if one can be obtained from the other through a series of Kempe changes.

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[1]Throughout this paper, all colourings are proper, i.e. no two vertices with the same colour are adjacent.
[2]If a vertex of G is coloured 1 and has no neighbour coloured 2 in α, then it forms a Kempe chain of size 1.

The study of Kempe changes has a vast history, see e.g. [13] for a comprehensive overview or [3] for a recent result on general graphs. We refer the curious reader to the relevant chapter of a 2013 survey by Cereceda [19]. Kempe equivalence falls within the wider setting of combinatorial reconfiguration, which [19] is also an excellent introduction to. Perhaps surprisingly, Kempe equivalence has direct applications in approximate counting and applications in statistical physics (see e.g. [15, 14] for nice overviews). Closer to graph theory, Kempe equivalence can be studied with a goal of obtaining a random colouring by applying random walks and rapidly mixing Markov chains, see e.g. [20].

Kempe changes were introduced as a mere tool, and are decisive in the proof of Vizing's edge colouring theorem [21]. However, the equivalence class they define on the set of k-colourings is itself highly interesting. In which cases is there a single equivalence class? In which cases does every equivalence class contain a colouring that uses the minimum number of colours? Vizing conjectured in 1965 [22] that the second scenario should be true in every line graph, no matter the choice of k. Despite partial results [1, 2], this conjecture remains wildly open.

In the setting of planar graphs, Meyniel proved in 1977 [12] that all 5-colourings form a unique Kempe equivalence class. The result was then extended to all K_5-minor-free graphs in 1979 by Las Vergnas and Meyniel [11]. They conjectured the following, which can be seen as a reconfiguration counterpoint to Hadwiger's conjecture, though it neither implies it nor is implied by it.

Conjecture 1.1 (Conjecture A in [11])**.** *For every t, all the t-colourings of a graph with no K_t-minor form a single equivalence class.*

They also proposed a related conjecture that is weaker assuming Hadwiger's conjecture holds.

Conjecture 1.2 (Conjecture A' in [11])**.** *For every t and every graph with no K_t-minor, every equivalence class of t-colourings contains some $(t-1)$-colouring.*

Here, we disprove both Conjectures 1.1 and 1.2, as follows.

Theorem 1.3. *For every $\varepsilon > 0$ and for any large enough t, there is a graph with no K_t-minor, whose $(\frac{3}{2} - \varepsilon)t$-colourings are not all Kempe equivalent.*

In fact, we prove that for every $\varepsilon > 0$ and for any large enough t, there is a graph G that does not admit a K_t-minor but admits a $(\frac{3}{2} - \varepsilon)t$-colouring that is *frozen*; Any pair of colours induce a connected component, so that no Kempe change can modify the colour partition. To obtain Theorem 1.3, we then argue that the graph admits a colouring with a different colour partition. The notion of frozen k-colouring is related to the notion of *quasi-K_p-minor*, introduced in [11]. A graph G admits a K_p-minor if it admits p non-empty, pairwise disjoint and connected *bags* $B_1, \dots, B_p \subset V(G)$ such that for any $i \neq j$, there is an edge between some vertex in B_i and some vertex in B_j. For the notion of quasi-K_p-minor, we drop the restriction that each B_i should induce a connected subgraph of G, and replace it with the condition that for any $i \neq j$, the set $B_i \cup B_j$ induces a connected subgraph of

G. If the graph G admits a frozen p-colouring, then it trivially admits a quasi-K_p-minor[3], while the converse may not be true. If all p-colourings of a graph form a single equivalence class, then either there is no frozen p-colouring or there is a unique p-colouring of the graph up to colour permutation. The latter situation in a graph with no K_p-minor would disprove Hadwiger's conjecture, so Las Vergnas and Meyniel conjectured that there is no frozen p-colouring in that case. Namely, they conjectured the following.

Conjecture 1.4 (Conjecture C in [11])**.** *For any t, any graph that admits a quasi-K_t-minor admits a K_t-minor.*

Conjecture 1.4 was proved for increasing values of t, and is now known to hold for $t \leq 10$ [5, 16, 10]. As discussed above, we strongly disprove Conjecture 1.4 for large t. It is unclear how large t needs to be for a counter-example.

Theorem 1.5. *For every $\varepsilon > 0$ and for any large enough t, there is a graph G that admits a quasi-K_t-minor but does not admit a $K_{(\frac{2}{3}+\varepsilon)t}$-minor.*

We later became aware a similar construction already appeared in [4].

Trivially, every graph that admits a quasi-K_{2t}-minor admits a K_t-minor. We leave the following two open questions, noting that $\frac{2}{3} \geq c \geq \frac{1}{2}$ and $c' \geq \frac{3}{2}$.

Question 1.6. What is the infimum c such that for any large enough t, there is a graph G that admits a quasi-K_t-minor but no K_{ct}-minor?

Question 1.7. Is there a constant c' such that for every t, all the $c' \cdot t$-colourings of a graph with no K_t-minor form a single equivalence class?

In the 1980's, [8, 9] and [17] proved independently that a graph with no K_t-minor has degeneracy $O(t\sqrt{\log t})$, since improved only by a constant factor [18, 23, 6]. Since all the k-colorings of d-degenerate graphs are equivalent for $k > d$ [11], this gives the best upper bound known so far for Question 1.7.

2 Construction

Let $n \in \mathbb{N}$ and let $\eta > 0$. We build a random graph G_n on vertex set $\{a_1, \dots, a_n, b_1, \dots, b_n\}$: for every $i \neq j$ independently, we select one pair uniformly at random among $\{(a_i, a_j), (a_i, b_j), (b_i, a_j), (b_i, b_j)\}$ and add the three other pairs as edges to the graph G_n.

Note that the sets $\{a_i, b_i\}_{1 \leq i \leq n}$ form a quasi-K_n-minor, as for every $i \neq j$, the set $\{a_i, b_i, a_j, b_j\}$ induces a path on four vertices in G_n, hence is connected.

Our goal is to argue that if n is sufficiently large then with high probability the graph G_n does not admit any $K_{(\frac{2}{3}+\eta)n}$-minor. This will yield Theorem 1.5. To additionally obtain Theorem 1.3, we need to argue that with high probability, G_n admits an n-colouring with a different colour partition than the natural one, where the colour classes are of the form $\{a_i, b_i\}$. Informally, we can observe that each of $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ induces a

[3] One bag for each colour class.

graph behaving like a graph in $\mathcal{G}_{n,\frac{3}{4}}$ (i.e. each edge exists with probability $\frac{3}{4}$) though the two processes are not independent. This argument indicates that $\chi(G_n) = O(\frac{n}{\log n})$, but we prefer a simpler, more pedestrian approach.

Assume that for some i, j, k, ℓ, none of the edges a_ib_j, a_jb_k, a_kb_ℓ and $a_\ell b_i$ exist. Then the graph G_n admits an n-colouring α where $\alpha(a_p) = \alpha(b_p) = p$ for every $p \notin \{i, j, k, \ell\}$ and $\alpha(a_i) = \alpha(b_j) = i$, $\alpha(a_j) = \alpha(b_k) = j$, $\alpha(a_k) = \alpha(b_\ell) = k$ and $\alpha(a_\ell) = \alpha(b_i) = \ell$ (see Figure 1). Since every quadruple (i, j, k, ℓ) has a positive and constant probability of satisfying this property, G_n contains such a quadruple with overwhelmingly high probability when n is large.

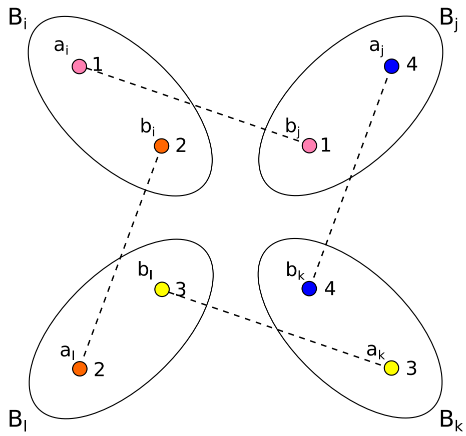

Figure 1: A different n-colouring given an appropriate quadruple.

We are now ready to prove that the probability that G_n admits a $K_{(\frac{2}{3}+\eta)n}$-minor tends to 0 as n grows to infinity. We consider three types of K_p-minors in G, depending on the size of the bags involved. If every bag is of size 1, we say that it is a *simple* K_p-minor – in fact, it is a subgraph. If every bag is of size 2, we say it is a *double* K_p-minor. If every bag is of size at least 3, we say it is a *triple* K_p-minor. We prove three claims, as follows.

Claim 2.1. *For any* $\varepsilon > 0$, $\mathbb{P}(G_n$ *contains a simple* $K_{\varepsilon n}$*-minor*$) \to 0$ *as* $n \to \infty$.

Claim 2.2. *For any* $\varepsilon > 0$, $\mathbb{P}(G_n$ *contains a double* $K_{\varepsilon n}$*-minor*$) \to 0$ *as* $n \to \infty$.

Claim 2.3. G_n *does not contain a triple* $K_{\frac{2}{3}n+1}$*-minor.*

Claims 2.1, 2.2 and 2.3 are proved in Sections 2.1, 2.2 and 2.3, respectively. If a graph admits a K_p-minor, then in particular it admits a simple K_a-minor, a double K_b-minor and a triple K_c-minor such that $a + b + c \geq p$. Combining Claims 2.1, 2.2 and 2.3, we derive the desired conclusion.

2.1 No large simple minor

Proof of Claim 2.1. Let S be a subset of k vertices of G_n. The probability that S induces a clique in G_n is at most $\left(\frac{3}{4}\right)^{\binom{k}{2}}$. Indeed, if $\{a_i, b_i\} \subseteq S$ for some i, then the probability is 0. Otherwise, $|S \cap \{a_i, b_i\}| \leq 1$ for every i, so we have $G[S] \in \mathcal{G}_{k,\frac{3}{4}}$, i.e. edges exist independently with probability $\frac{3}{4}$. Therefore, the probability that S induces a clique is $\left(\frac{3}{4}\right)^{\binom{k}{2}}$. By union-bound, the probability that some subset on k vertices induces a clique is at most $\binom{2n}{k} \cdot \left(\frac{3}{4}\right)^{\binom{k}{2}}$. For any $\varepsilon > 0$, we note that $\binom{2n}{\varepsilon n} \leq 2^{2n}$. Therefore, the probability that G_n contains a simple $K_{\varepsilon n}$-minor is at most $2^{2n} \cdot \left(\frac{3}{4}\right)^{\binom{\varepsilon n}{2}}$, which tends to 0 as n grows to infinity. ⌟

2.2 No large double minor

Proof of Claim 2.2. Let S' be a subset of k pairwise disjoint pairs of vertices in G_n such that for every i, at most one of $\{a_i, b_i\}$ is involved in S'.

We consider the probability that $G_n/_{S'}$ induces a clique, where $G_n/_{S'}$ is defined as the graph obtained from G_n by considering only vertices involved in some pair of S' and identifying the vertices in each pair.

We consider two distinct pairs $(x_1, y_1), (x_2, y_2)$ of S'. Without loss of generality, $\{x_1, x_2, y_1, y_2\} = \{a_i, a_j, a_k, a_\ell\}$ for some i, j, k, ℓ. The probability that there is an edge between $\{x_1, y_1\}$ and $\{x_2, y_2\}$ is $1 - \left(\frac{1}{4}\right)^4$. In other words, $\mathbb{P}(E((x_1, y_1), (x_2, y_2)) = \emptyset) = \left(\frac{1}{4}\right)^4$ and since at most one of $\{a_i, b_i\}$ is involved in S' for all i, all such events are mutually independent. Therefore, the probability that S' yields a quasi-$K_{|S'|}$-minor is $\left(1 - \left(\frac{1}{4}\right)^4\right)^{\binom{|S'|}{2}}$.

For any $\varepsilon' > 0$, the number of candidates for S' is at most $\binom{2n}{2\varepsilon' n}$ (the number of choices for a ground set of $2\varepsilon' n$ vertices) times $(2\varepsilon' n)!$ (a rough upper bound on the number of ways to pair them). Note that $\binom{2n}{2\varepsilon' n} \cdot (2\varepsilon' n)! \leq (2n)^{2\varepsilon' n}$. We derive that the probability that there is a set S' of size $\varepsilon' n$ such that $G_n/_{S'} = K_{|S'|}$ is at most $(2n)^{2\varepsilon' n} \cdot \left(1 - \left(\frac{1}{4}\right)^4\right)^{\binom{\varepsilon' n}{2}}$, which tends to 0 as n grows large.

Consider a double K_k-minor S of G_n. Note that no pair in S is equal to $\{a_i, b_i\}$ (for any i), as every bag induces a connected subgraph in G_n. We build greedily a maximal subset $S' \subseteq S$ such that S' involves at most one vertex out of every set of type $\{a_i, b_i\}$. Note that $|S'| \geq \frac{|S|}{3}$. By taking $\varepsilon' = \frac{\varepsilon}{3}$ in the above analysis, we obtain that the probability that there is a set S of εn pairs that induces a quasi-$K_{|S|}$-minor tends to 0 as n grows large. ⌟

2.3 No large triple minor

Proof of Claim 2.3. The graph G_n has $2n$ vertices, and a triple K_k-minor involves at least $3k$ vertices. It follows that if G_n contains a triple K_k-minor then $k \leq \frac{2n}{3}$. ⌟

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