Big Ramsey Degrees in the Metric Setting

(Extended abstract)

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Abstract

Oscillation stability is an important concept in Banach space theory which happens to be closely connected to discrete Ramsey theory. For example, Gowers proved oscillation stability for the Banach space c_0 using his now famous Ramsey theorem for FIN_k as the key ingredient. We develop the theory behind this connection and introduce the notion of compact big Ramsey degrees, extending the theory of (discrete) big Ramsey degrees. We prove existence of compact big Ramsey degrees for the Banach space ℓ_∞ and the Urysohn sphere, with an explicit characterization in the case of ℓ_∞.

1   Introduction

A *discrete structure* is a structure in a relational language without unary predicates (e.g. orders, graphs, hypergraphs, etc.). Given two discrete structures X and Y in the same language we will denote by $\binom{X}{Y}$ the set of all embeddings $Y \to X$. Given a discrete structure X and a finite substructure A of X we say that *A has a finite big Ramsey degree* in X if there exists $t \geqslant 1$ such that every finite colouring of $\binom{X}{A}$ attains at most t colors

on $\binom{f[X]}{A}$ for some well-chosen $f \in \binom{X}{X}$. In this case, the *big Ramsey degree* of A in X is the least such t. We say that X *has finite big Ramsey degrees* if every finite substructure of X has a finite big Ramsey degree in X.

The infinite Ramsey theorem says that all big Ramsey degrees in $(\mathbb{N}, <)$ are equal to 1. However, this is usually not the case. By Hjorth [10], no infinite homogeneous structure has all big Ramsey degrees equal to 1. However, they can still be finite. The task of identifying such structures was initiated by an unpublished result by Laver who proved finiteness of big Ramsey degrees of $(\mathbb{Q}, <)$, followed by their exact computation by Devlin [5]. Since then there has been a lot of progress [1, 2, 3, 4, 6, 8, 11, 13, 15, 16, 21]. See also Dobrinen's survey [7].

Recall that c_0 is the Banach space of all sequences $x \colon \mathbb{N} \to \mathbb{R}$ tending to zero at infinity, with the norm $||x||_\infty := \sup_{n \in \mathbb{N}} |x(n)|$, and that the *unit sphere* of a Banach space X is $S_X := \{x \in X \mid ||x|| = 1\}$. Also recall that the *Urysohn sphere* $\mathbb{S}$ is the unique complete and separable metric space of diameter 1 containing copies of all finite metric spaces of diameter at most 1 which is *homogeneous*, that is, every isometry between finite subsets of $\mathbb{S}$ extends to an onto isometry of $\mathbb{S}$. By a *compactum*, we mean a compact metric space.

A discrete structure X is *indivisible* if the big Ramsey degree of a vertex in X is equal to 1. Mostly motivated by the *distortion problem* from Banach space theory, the two first indivisibility-like results for metric structures (called *oscillation stability results* by Banach space theorists) have been proved by Gowers [9], and Nguyen Van Thé and Sauer [17]. Here, Lipschitz maps can be seen as continuous colourings, and compactness is the right metric analogue of finiteness.

Theorem 1.1 (Gowers [9]). *Let K be a compactum and $\chi \colon S_{c_0} \to K$ be a Lipschitz map. For every $\varepsilon > 0$, there exists a linear isometric copy X of c_0 in c_0 such that* $\mathrm{diam}(\chi(S_X)) \leqslant \varepsilon$.

Theorem 1.2 (Nguyen Van Thé–Sauer [17]). *Let K be a compactum and $\chi \colon \mathbb{S} \to K$ be a Lipschitz map. For every $\varepsilon > 0$, there exists an isometric copy X of $\mathbb{S}$ in $\mathbb{S}$ such that* $\mathrm{diam}(\chi(X)) \leqslant \varepsilon$.

The proof of Theorem 1.1 is based on discrete approximations and was the reason why Gowers proved his now well-known FIN_k theorem, which is the main ingredient in the proof. The proof of Theorem 1.2 is also using discrete approximations (following a combinatorial strategy which was proposed earlier by Nguyen Van Thé and Lopez-Abad [14]) combined with indivisibility results for metric spaces with finitely many distances.

The similarity between those results and indivisibility results makes it is natural to ask if a suitable version of big Ramsey degrees could be defined for metric structures. This was addressed by Kechris, Pestov, and Todorcevic [12, § 11(F)] who suggested a definition which, however, is rather restrictive and fails to capture most interesting structures beyond the discrete ones. Our goal is to provide a more general notion, and to demonstrate its suitability on examples such as the Banach space ℓ_∞ and the Urysohn sphere. Our work on the Urysohn sphere builds on results on big Ramsey degrees of homogeneous structures with forbidden cycles announced at Eurocomb 2021 [3].

Our motivations are twofold. First, Zucker [20] extended the KPT correspondence [12] to big Ramsey degrees, giving a correspondence between them and some dynamical invariants of automorphism groups. Our extension to the metric setting could allow us to study the same dynamical invariants for the automorphism groups of metric structures; no tool is currently available for studying those. Second, our methods could lead to a systematical study of the distortion phenomenon in Banach space theory, closely related to oscillation stability and not yet well understood. For instance, Odell and Schlumprecht's solution to the distortion problem [18] shows that the analogue of Theorem 1.1 fails for the separable Hilbert space. Metric big Ramsey degrees could help to express a quantitative and optimal version of their result.

2 Compact big Ramsey degrees

We first review, in a more general setting, some results on discrete big Ramsey degrees to motivate our definitions in the metric case. Let M be a monoid acting by injections on a set X. The action $M \curvearrowright X$ *has a finite big Ramsey degree* if there exists $t \geqslant 1$ such that every colouring of X with finitely many colours takes at most t values on a set of the form $p \cdot X$, $p \in M$. In this case, the *big Ramsey degree* of the action is the least such t. Observe that if Y is a discrete structure and $A \subseteq Y$ a finite substructure, then taking $M := \binom{Y}{Y}$ and $X := \binom{Y}{A}$ and considering the action by left-composition, we recover the classical notion of the big Ramsey degree of A in Y. For $k \geqslant 1$, denote by $[k]$ the set $\{1, \ldots, k\}$.

Definition 2.1. Fix $M \curvearrowright X$ as above, and $k \geqslant 1$. A colouring $\chi \colon X \to [k]$ is:

- *persistent* if for every $p \in M$, $\chi(p \cdot X) = [k]$;

- *universal* if for every $l \geqslant 1$, every colouring $\psi \colon X \to [l]$ and every $p \in M$, there exists $q \in M$ and $f \colon [k] \to [l]$ such that $\psi \upharpoonright_{pq \cdot X} = f \circ \chi \upharpoonright_{pq \cdot X}$;

- a *big Ramsey colouring* (or a *canonical partition*, following [13]) if it is both persistent and universal.

The proof of the following fact is elementary.

Proposition 2.2. *Suppose that the action $M \curvearrowright X$ has a finite big Ramsey degree. Then it admits a big Ramsey colouring. Moreover, the number of colours of such a colouring is always equal to the big Ramsey degree of the action.*

In the metric metric setting, one has a monoid M acting by (non-necessarily onto) isometries on a complete metric space X. Inspired by Theorems 1.1 and 1.2, we define a *colouring* of X as a 1-Lipschitz map $X \to K$, where K is a compactum (the Lipschitz constant 1 is here to ensure some rigidity). We will also allow some ε-approximation in our results. For K, L compacta we put $K \leqslant L$ if there exists a 1-Lispchitz surjection $L \to K$. This quasiordering is meant to "replace" the order on $\mathbb{N}$. It is a classical fact that $K \leqslant L$ and $L \leqslant K$ if and only if K and L are isometric. If $\chi, \psi \colon X \to K$ are maps, we will denote by $d_\infty(f, g)$ the supremum distance between f and g.

Definition 2.3. Say that a compactum K is:

- *universal* with respect to the action $M \curvearrowright X$ if for every compactum L, every colouring $\psi \colon X \to L$, and every $\varepsilon > 0$, there exists $q \in M$, a colouring $\chi \colon X \to K$, and a 1-Lipschitz map $f \colon K \to L$ such that $d_\infty(\psi \restriction_{q \cdot X}, f \circ \chi \restriction_{q \cdot X}) \leqslant \varepsilon$;

- the *big Ramsey degree of the action* $M \curvearrowright X$ if K is a $\leqslant$-least universal compactum.

Say that the action $M \curvearrowright X$ *has a compact big Ramsey degree* if it admits a big Ramsey degree in the above sense.

The big Ramsey degree of an action, if it exists, is unique, up to isometry.

Definition 2.4. Say that a colouring $\chi \colon X \to K$ is:

- *persistent* if for every $p \in M$, $\chi(p \cdot X)$ is dense in K;

- *universal* if for every compactum L, every colouring $\psi \colon X \to L$, every $p \in M$ and every $\varepsilon > 0$, there exists $q \in M$ and a 1-Lipschitz map $f \colon K \to L$ such that $d_\infty(\psi \restriction_{pq \cdot X}, f \circ \chi \restriction_{pq \cdot X}) \leqslant \varepsilon$;

- a *big Ramsey colouring* if it is both persistent and universal.

Proposition 2.5. *Suppose that* $\chi \colon X \to K$ *is a big Ramsey colouring for the action* $M \curvearrowright X$. *Then* K *is the big Ramsey degree of this action.*

Proposition 2.6. *Consider the following statements:*

(1) the action $M \curvearrowright X$ admits a universal compactum;

(2) the action $M \curvearrowright X$ has a compact big Ramsey degree;

(3) the action $M \curvearrowright X$ admits a universal colouring;

(4) the action $M \curvearrowright X$ admits a big Ramsey colouring.

Then the following implications hold: (4) $\implies$ (3) $\implies$ (2) $\implies$ (1).

While the analogues of the above implications are equivalent in the discrete setting, we do not know whether any of the reverse implications holds in the metric setting. The most relevant notion seems to be the existence of a big Ramsey colouring as, in the discrete setting, it is the closest to Zucker's condition for getting interesting dynamical consequences [20]. Also, in all metric examples for which we have been able to prove the existence of a universal compactum, we could also prove the existence of a big Ramsey colouring.

We end this section mentioning that, by endowing discrete structures with the metric where any two distinct points are at distance 1, we can "embed" the classical discrete setting for big Ramsey degrees in our metric setting, thus making the discrete setting a particular case of the metric setting.

3 Banach spaces

In this section we study big Ramsey degrees of the spaces ℓ_p and c_0. Instead of colouring (linear isometric) embeddings of finite-dimensional subspaces into the whole space, we will equivalently colour finite tuples of elements of its unit sphere, which makes the presentation easier. Given a Banach space X and $d \geqslant 1$, the set $(S_X)^d$ will be endowed with the supremum distance. We denote by $\mathrm{Emb}(X)$ the monoid of all linear isometric embeddings of X into itself.

For $1 \leqslant p < \infty$ and a sequence $x \colon \mathbb{N} \to \mathbb{R}$, we let $\|x\|_p := \left(\sum_{n\in\mathbb{N}} |x(n)|^p\right)^{\frac{1}{p}}$, and $\|x\|_\infty := \sup_{n\in\mathbb{N}} |x(n)|$; and for $1 \leqslant p \leqslant \infty$, we denote by ℓ_p the Banach space of all sequences $x \colon \mathbb{N} \to \mathbb{R}$ such that $\|x\|_p < \infty$, endowed with the norm $\|\cdot\|_p$. The space c_0 is a particular subspace of ℓ_∞.

Gowers' theorem 1.1 is equivalent to saying that the action $\mathrm{Emb}(c_0) \curvearrowright S_{c_0}$ admits a big Ramsey degree which is a singleton. However, the situation is different in higher arities. To see this, colour a pair $(x,y) \in S_{c_0}^2$ of disjointly supported vectors by the number of times their supports intertwine. Then every block-subspace of c_0 meets an infinite number of colours. While this "colouring" is neither Lipschitz nor compactum-valued, the idea can be developed to prove the following result.

Theorem 3.1. *Let $d \geqslant 2$. Then the action $\mathrm{Emb}(c_0) \curvearrowright (S_{c_0})^d$ does not admit a universal compactum.*

As mentioned in the introduction, Odell and Schlumprecht [18] proved that the separable Hilbert space ℓ_2 does not satisfy an analogue of Theorem 1.1. In fact, their paper immediately implies a stronger conclusion.

Theorem 3.2. *Let $d \geqslant 1$ and $1 \leqslant p < \infty$. Then the action $\mathrm{Emb}(\ell_p) \curvearrowright (S_{\ell_p})^d$ does not admit a universal compactum.*

This means that for an optimal version of the Odell–Schlumprecht theorem (if it exists), a theory of noncompact big Ramsey degrees would be needed; we believe that such a theory can be developed, and keep it in mind for a future project.

We now turn to ℓ_∞. In the rest of this section, we fix $d \geqslant 1$ and consider the action $\mathrm{Emb}(\ell_\infty) \curvearrowright (S_{\ell_\infty})^d$. Classical arguments from Banach space theory show that, even when $d = 1$, this action does not admit a universal compactum. However, the proof involves a diagonal argument based on the Axiom of Choice. In such cases, imposing a definability restriction on colourings often allows one to get positive results (see e.g. [19]). The right topology is the *weak-* topology* (here, we refer to the one we get when seeing ℓ_∞ as the dual of ℓ_1). We can then define the notions of a *definable big Ramsey degree* and a *definable big Ramsey colouring* for the above action by considering, in Definitions 2.3 and 2.4, only colourings that are Borel, or even Suslin–measurable, when $(S_{\ell_\infty})^d$ is endowed with the d-th power of the weak-* topology. All results stated in Section 2 remain valid for these definable notions, and it turns out that we can prove the existence of a definable big Ramsey colouring for the action $\mathrm{Emb}(\ell_\infty) \curvearrowright (S_{\ell_\infty})^d$. In order to state our result, preliminary definitions are needed.

Put $B_d := [-1,1]^d$, endowed with the supremum metric. The entries of a tuple $x \in (S_{\ell_\infty})^d$ will be denoted by $x_1, \ldots, x_d$. We use a functional notation for elements of ℓ_∞, so that for $n \in \mathbb{N}$ and $i \in [d]$, the n-th entry of the vector x_i will be denoted by $x_i(n)$. We can finally let, for each $n \in \mathbb{N}$, $x(n)$ be the d-tuple $(x_1(n), \ldots, x_d(n))$; it is an element of B_d. In this way, elements of $(S_{\ell_\infty})^d$ can be seen as maps $\mathbb{N} \to B_d$. If X is a metric space, denote by $\mathcal{K}(X)$ the set of all nonempty compact subsets of X, and endow it with the *Hausdorff metric* d_H defined by $d_H(K, L) := \max(\sup_{x \in K} d(x, L), \sup_{y \in L} d(y, K))$. Denote by $\mathcal{SCK}(B_d)$ the set of all nonempty symmetric, convex and compact subsets of B_d, and see it as a metric subspace of $\mathcal{K}(B_d)$. If $A \subseteq B_d$, denote by $\mathrm{sc}(A)$ the symmetric convex hull of the set A.

Definition 3.3. A *d-pumpkin* is a compact subset $\mathcal{P} \subseteq \mathcal{SCK}(B_d)$ such that $\{0\} \in \mathcal{P}$, there exists $C \in \mathcal{P}$ such that for all $i \in [d]$, $\mathrm{proj}_i(C) = [-1, 1]$, and the inclusion induces a dense linear order on $\mathcal{P}$. We denote by Pum_d the set of all d-pumpkins, seen as a subset of $\mathcal{K}(\mathcal{SCK}(B_d))$.

A d-pumpkin can be seen as a continuously growing symmetric compact convex subset of B_d, starting at $\{0\}$ and such that the final step of the evolution touches all faces of the cube B_d. It can be shown that the metric space Pum_d is compact.

Definition 3.4. For $x \in (S_{\ell_\infty})^d$, let:

$$\mathrm{PP}_d(x) := \left\{ \mathrm{sc}\{x(0), \ldots, x(n-1), tx(n)\} \,\middle|\, n \in \mathbb{N}, t \in [0,1] \right\} \cup \left\{ \overline{\mathrm{sc}\{x(n) \mid n \in \mathbb{N}\}} \right\}.$$

This defines a definable colouring $\mathrm{PP}_d := (S_{\ell_\infty})^d \to \mathrm{Pum}_d$.

Intuitively, sets $\mathrm{sc}\{x(0), \ldots, x(n-1)\}$, $n \in \mathbb{N}$, must be steps of the evolution of the pumpkin $\mathrm{PP}_d(x)$, and the set $\overline{\mathrm{sc}\{x(n) \mid n \in \mathbb{N}\}}$ must be its final step. Between those steps, we "fill in the holes" in an affine way.

Theorem 3.5. PP_d *is a definable big Ramsey colouring of the action* $\mathrm{Emb}(\ell_\infty) \curvearrowright (S_{\ell_\infty})^d$. *In particular,* Pum_d *is the definable big Ramsey degree of this action.*

It is easy to see that Pum_1 is a singleton. Thus, as a corollary of Theorem 3.5, we get the following oscillation stability result for ℓ_∞, analogous to Theorem 1.1.

Corollary 3.6. *Let K be a compactum and $\chi \colon S_{\ell_\infty} \to K$ be a Lipschitz map that is also Borel (or Suslin-measurable) for the weak-* topology. Then for every $\varepsilon > 0$, there exists a linear isometric copy X of ℓ_∞ in itself such that* $\mathrm{diam}(\chi(S_X)) \leqslant \varepsilon$.

The proof of Theorem 3.5 is based on the Carlson–Simpson theorem. The natural presentation of S_{ℓ_∞} as a set of infinite words over the alphabet $[-1, 1]$ makes its use particularly simple. Another ingredient in the proof is an analysis of the form of linear isometric copies of ℓ_∞ in itself, based on elementary Banach space theoretic tools.

4   The Urysohn sphere

Recall that $\mathbb{S}$ is the Urysohn sphere. As for Banach spaces, we will consider colourings of tuples from $\mathbb{S}$ rather than embeddings of finite substructures. For each $d \geqslant 1$, endow the d-th power $\mathbb{S}^d$ with the supremum metric. Denote by $\mathrm{Emb}(\mathbb{S})$ the monoid of all (non-necessarily surjective) isometries of $\mathbb{S}$ into itself.

Theorem 4.1. *For every $d \geqslant 1$, the action $\mathrm{Emb}(\mathbb{S}) \curvearrowright \mathbb{S}^d$ admits a big Ramsey colouring.*

Our proof method is based on ideas developed in [3] for proving finiteness of the big Ramsey degrees of discrete versions of the Urysohn sphere. We don't work directly on $\mathbb{S}$ itself but on a metric space $\mathbb{T}$ that is bi-embeddable with it. This metric space is a well-enough behaved space of sequences, allowing us the use of the Carlson–Simpson theorem. Our proof allows us to recover the fact that the big Ramsey degree of the action $\mathrm{Emb}(\mathbb{S}) \curvearrowright \mathbb{S}$ is a singleton, thus giving a new and short proof of Theorem 1.2, based on very different tools than the original proof. However, as soon as $d > 1$, part of our proof relies on a non-constructive argument, and we are currently not able to characterize the big Ramsey degree completely. We are only able to give an upper bound of the big Ramsey degree in the sense of the quasiordering $\leqslant$, as a quotient of $\mathbb{T}^d$ by an action of the monoid of rigid surjections $\mathbb{N} \to \mathbb{N}$.

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