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Chromatic number of intersection graphs of segments with two slopes*

(Extended abstract)

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Abstract

A *d-dir* graph is an intersection graph of segments, where the segments have at most d different slopes. It is easy to see that a d-dir graph with clique number ω has chromatic number at most $d\omega$. We study the chromatic number of 2-dir graphs in more detail, proving that this upper bound is tight even in the fractional coloring setting.

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The clique number (the size of the largest complete subgraph) is clearly a lower bound on the chromatic number of a graph. It is natural to ask whether the chromatic number can be also bounded from above as a function of the clique number. In particular, a graph G is called *perfect* if G as well as all its induced subgraphs have chromatic number equal to clique number. Graphs from many interesting graph classes, such as bipartite graphs, their linegraphs, interval and chordal graphs, and their complements, are known to be perfect. A celebrated strong perfect graph theorem [5] states that a graph is perfect if and only if it avoids odd holes and antiholes.

Relaxing the notion of perfectness, Gyárfás [11] introduced the notion of χ-boundedness: A graph class $\mathcal{G}$ is χ*-bounded* if there exists a function f such that $\chi(G) \le f(\omega(G))$ for every $G \in \mathcal{G}$. Not all graphs are χ-bounded—Erdős [8] famously proved that there exist graphs with arbitrarily large chromatic number and arbitrarily large girth. However, many interesting graph classes have this property. For example, in the geometric setting,

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intersection graphs of scaled and translated copies of any compact convex shape [13], circle graphs (intersection graphs of the chords of a circle) [10, 6], intersection graphs of unit-length segments in the plane [18] and intersection graphs of axis-aligned rectangles in the plane [1] are χ-bounded (on the other hand, triangle-free intersection graphs of boxes in $\mathbb{R}^3$ have unbounded chromatic number [2]).

In this paper, we are interested in a variant of intersection graphs of straight line segments in the plane. A graph G is a *segment intersection graph* if we can assign to each vertex of G a segment in the plane so that distinct vertices $u, v \in V(G)$ are adjacent iff the corresponding segments intersect in at least one point. Answering in negative an old question of Erdős, Pawlik et al. [16] proved that there exist triangle-free segment intersection graphs with arbitrarily large chromatic number. However, it is easy to see that such graphs must contain segments of many different slopes. Indeed, suppose each segment of a segment intersection graph G has one of at most d slopes (such graphs are called *d-dir*). The segments in each direction induce an interval graph, and since interval graphs are perfect, we conclude that $\chi(G) \leq d \cdot \omega(G)$. Hence, for every d, the class of d-dir graphs is χ-bounded.

For context, all planar graphs are segment intersection graphs [4], bipartite planar graphs are 2-dir [7, 12], and 3-colorable planar graphs are 3-dir [9]. Moreover, West [19] conjectured all planar graphs are 4-dir. Let us remark that recognizing whether a graph is a d-dir graph is NP-complete for any $d \geq 2$, while recognizing the segment intersection graphs in general is $\exists\mathbb{R}$-complete, and thus NP-hard and known to be in PSPACE [14].

There are very few χ-bounded graph classes where tight bounds on the chromatic number in terms of the clique number are known. This motivates us to ask whether the trivial bound on the chromatic number of d-dir graphs can be improved. In this paper, we focus on the basic case $d = 2$. Without loss of generality, each 2-dir graph G is represented by horizontal and vertical segments. A *row* of G is a horizontal line that contains at least one of the horizontal segments of G. To get a more detailed understanding, we consider 2-dir graphs with bounded number of rows: Let $\mathcal{D}_{k,t}$ denote the class of all 2-dir graphs of clique number at most t and with at most k rows, and let $\mathcal{D}_t = \bigcup_{k=1}^{\infty} \mathcal{D}_{k,t}$ be the class of all 2-dir graphs of clique number at most t.

It turns out to be convenient to consider *fractional chromatic number* in addition to standard chromatic number. A *fractional coloring* of a graph G is a function φ that to each vertex assigns a set of measure one (in any measure space), such that $\varphi(u) \cap \varphi(v) = \emptyset$ for every $uv \in E(G)$. The *span* of φ is the measure of $\bigcup_{v \in V(G)} \varphi(v)$. The *fractional chromatic number* $\chi_f(G)$ of G is the infimum of the real numbers c such that G has a fractional coloring of span at most c. It is known that the infimum is actually a minimum and that it is always rational [17]. Clearly $\chi_f(G) \leq \chi(G)$ for every graph G. Moreover, it is known that there exist graphs with fractional chromatic number arbitrarily close to 2 and with arbitrarily large chromatic number [15].

For a class of graphs $\mathcal{G}$, let $\chi(\mathcal{G})$ and $\chi_f(\mathcal{G})$ denote the supremum of chromatic and fractional chromatic numbers, respectively, of graphs from $\mathcal{G}$. Our results are based on a detailed investigation of the (fractional) chromatic number of triangle-free 2-dir graphs with a given number of rows, summarized in the following theorem.

Theorem 1. *The following bounds hold:*

- $\chi(\mathcal{D}_{1,2}) = \chi_f(\mathcal{D}_{1,2}) = 2$.
- $\chi(\mathcal{D}_{2,2}) = 3$ *and* $\chi_f(\mathcal{D}_{2,2}) = 5/2$.
- $\chi(\mathcal{D}_{3,2}) = 3$ *and* $\chi_f(\mathcal{D}_{3,2}) = 11/4$.
- $\chi(\mathcal{D}_{k,2}) = 4$ *for every* $k \geq 4$ *and* $\chi_f(\mathcal{D}_{4,2}) = 3$.
- $\chi_f(\mathcal{D}_{k,2}) \leq 4 - \frac{1}{2^{k-1}}$ *for every* $k \geq 1$.
- *For all integers* $r, m \geq 1$, $\chi_f(\mathcal{D}_{rm,2}) \geq 4 - \frac{1}{m} - \frac{1}{2^{r-1}}$.

Consequently, $\chi(\mathcal{D}_2) = \chi_f(\mathcal{D}_2) = 4$*, but there are no triangle-free 2-dir graphs with fractional chromatic number exactly* 4*.*

Let us remark that the fact that $\chi_f(\mathcal{D}_2) = 4$ also follows from the construction of [3], who gave construction of 2-dir triangle-free graphs G with $|V(G)|/\alpha(G)$ arbitrarily close to 4.

For a positive integer b, the *b-blowup* of a graph G is the graph H obtained from G by replacing each vertex by a clique of size b and each edge uv by a complete bipartite graph between the cliques replacing u and v. Note that clique number, fractional chromatic number, and being a d-dir graph all behave predictably after a blowup (this is not the case for the ordinary chromatic number, motivating our focus on its fractional version): We have $\omega(H) = b \cdot \omega(G)$ and $\chi_f(H) = b \cdot \chi_f(G)$, and if G is a d-dir graph, then H is a d-dir graph as well.

Since $\chi_f(\mathcal{D}_2) = 4$, for every even integer t and $\varepsilon > 0$, there exists a triangle-free 2-dir graph with fractional chromatic number at least $4 - \frac{2\varepsilon}{t}$, and applying the $(t/2)$-blowup operation results in a graph of clique number t and fractional chromatic number at least $2t - \varepsilon$. This gives our main result: For 2-dir graphs, the trivial upper bound $\chi(\mathcal{D}_t) \leq 2t$ cannot be improved when the clique number t is even.

Corollary 2. *For every even* t*, we have* $\chi(\mathcal{D}_t) = \chi_f(\mathcal{D}_t) = 2t$*.*

For odd t, this only gives a bound $2t - 2 \leq \chi_f(\mathcal{D}_t) \leq \chi(\mathcal{D}_t) \leq 2t$. We suspect the upper bound is tight in this case as well. Finally, we conjecture this is the case for d-dir graphs in general.

Conjecture 3. *For all positive integers* d *and* t *and real* $\varepsilon > 0$*, there exists a* d*-dir graph* G *of clique number* t *whose fractional chromatic number is at least* $dt - \varepsilon$*.*

Because of the blowup operation, to prove this for even t one only needs to consider the case $t = 2$, i.e., triangle-free d-dir graphs. In the rest of this extended abstract, let us outline the construction showing the lower bounds on the fractional chromatic number stated in Theorem 1.

Lower bounds on the fractional chromatic number

Throughout the rest of the paper, let μ be Lebesgue measure on subsets of real numbers. We will work with a slightly more general notion of fractional coloring. Let G be a graph and let $f : V(G) \to \mathbb{R}_0^+$ be an arbitrary function. An *f-fractional coloring* of G is a function φ that to each vertex $v \in V(G)$ assigns a set of real numbers such that $\mu(\varphi(v)) = f(v)$ and $\varphi(u) \cap \varphi(v) = \emptyset$ for every $uv \in E(G)$. The *f-fractional chromatic number* of G is the infimum of the spans of its f-fractional colorings.

We say that a 2-dir graph is *horizontally trivial* if each of its rows contains exactly one horizontal segment v and this segment v intersects all vertical segments that intersect the row. In other words, the segment v can be extended arbitrarily far in each direction along the row without changing the intersection graph. The motivation for requiring different amounts of colors at each vertex comes from the following observation.

Lemma 4. *Let $r, m \geq 1$ be integers and let G be a horizontally trivial triangle-free 2-dir graph with at most m rows. Let $f : V(G) \to \mathbb{R}^+$ be defined by setting $f(v) = 1$ for each vertical segment v and $f(v) = 2 - \frac{1}{2^{r-1}}$ for each horizontal segment v. Then G has f-fractional chromatic number at most $\chi_f(\mathcal{D}_{rm,2})$.*

Proof. Without loss of generality, we can assume that for $i \in \{0, \ldots, m-1\}$, G has a horizontal segment h_i with endpoints $(0, i)$ and $(1, i)$, and that each vertical segment has the x-coordinate strictly between 0 and 1. Moreover, we can assume that for each endpoint $p = (x, y)$ of a vertical segment, there exists an integer j such that $j - 1/r < y < j$: For any integer j, we can shift all endpoints with $j - 1 < y < j$ to this interval without changing the intersection graph; and if an endpoint p has y-coordinate exactly j, then since G is triangle-free, it is an endpoint of exactly one vertical segment v, and thus we can shift p so that its y-coordinate becomes less than j (if p is the bottom endpoint of v) or more than $j + 1 - 1/r$ (if p is the top endpoint of v).

Let G' be the triangle-free 2-dir graph with rm rows obtained as follows: We copy each vertical segment of G 2^{rm-1} times and shift the copies by 0, 1, …, $2^{rm-1} - 1$ to the right. For $i = 0, \ldots, rm-1$, G' has a row ℓ_i with y-coordinate $y_i = i/r$, and this row contains 2^i horizontal segments $v_{i,j}$ for $j \in \{0, \ldots, 2^i - 1\}$, where $v_{i,j}$ has endpoints $(j \cdot 2^{rm-1-i}, y_i)$ and $((j+1) \cdot 2^{rm-1-i}, y_i)$. We view the horizontal segments as arranged in a tree-like fashion, and if $i < rm - 1$, we say that $v_{i+1,2j}$ and $v_{i+1,2j+1}$ are the *children* of $v_{i,j}$; note that the projection of $v_{i,j}$ on the x-axis is the union of the projections of its children. Note also that by the assumptions on the y-coordinates of the endpoints of the vertical segments of G, for $k \in \{0, \ldots, m-1\}$, the rows ℓ_{kr}, ℓ_{kr+1}, …, ℓ_{kr+r-1} intersect exactly the same vertical segments of G'.

Since $G' \in \mathcal{D}_{rm,2}$, G' has a fractional coloring φ of span at most $\chi_f(\mathcal{D}_{rm,2})$. For $k = 1, \ldots, m$, let us choose a horizontal segment $u_k = v_{kr-1,j_k}$ in the row ℓ_{kr-1} and a set C_k of colors of measure $2 - \frac{1}{2^{r-1}}$ as follows:

- If $k = 1$, then let $u_{k,0} = v_{0,0}$, otherwise let $u_{k,0}$ be a child of u_{k-1} chosen arbitrarily, and let $C_{k,0} = \varphi(u_{k,0})$.

- For $i = 1, \ldots, r-1$, choose $u_{k,i} = u'$ among the children u' and u'' of $u_{k,i-1}$ so that $\mu(C_{k,i-1} \cap \varphi(u')) \leq \mu(C_{k,i-1} \cap \varphi(u''))$. Since $u'u'' \in E(G')$, $\varphi(u')$ and $\varphi(u'')$ are disjoint, and thus $\mu(C_{k,i-1} \cap \varphi(u')) \leq \frac{1}{2}\mu(C_{k,i-1})$. Hence, $\mu(C_{k,i-1} \cup \varphi(u')) = \mu(C_{k,i-1}) + 1 - \mu(C_{k,i-1} \cap \varphi(u')) \geq \frac{1}{2}\mu(C_{k,i-1}) + 1$. We let $C_{k,i}$ be a subset of $C_{k,i-1} \cup \varphi(u')$ of measure exactly $2 - \frac{1}{2^i}$. Such a set exists, since by induction, we have $\frac{1}{2}\mu(C_{k,i-1}) + 1 = \frac{1}{2}\big(2 - \frac{1}{2^{i-1}}\big) + 1 = 2 - \frac{1}{2^i}$.
- Let $u_k = u_{k,r-1}$ and $C_k = C_{k,r-1}$.

Consider now the assignment φ' of sets of colors to vertices of G defined as follows: For $k = 1, \ldots, m$, the horizontal segment h_{k-1} with endpoints $(0, k-1)$ and $(1, k-1)$ gets $\varphi'(h_{k-1}) = C_k$. Hence, $\mu(\varphi'(h_{k-1})) = 2 - \frac{1}{2^{r-1}} = f(h_{k-1})$. For each vertical segment v, let v' be its copy in G' whose projection on the x-axis is contained inside the projection of u_m (and thus also inside the projection of $u_1, \ldots, u_{m-1}$), and let $\varphi'(v) = \varphi(v')$. Note that if v' intersects u_k for some $k \in \{1, \ldots, m\}$, then it also intersects all horizontal segments whose color sets contribute to C_k, and thus $\varphi'(v)$ is disjoint from $C_k = \varphi'(h_{k-1})$. Consequently, φ' is an f-fractional coloring of G whose span is at most the span of φ, and thus at most $\chi_f(\mathcal{D}_{rm,2})$ □

We use duality to prove lower bounds on the fractional chromatic number. A function $\gamma : V(G) \to \mathbb{R}_0^+$ is a *fractional clique* in a graph G if $\sum_{v \in I} \gamma(v) \leq 1$ for every independent set I in G. The *f-weight* of γ is defined as $\sum_{v \in V(G)} \gamma(v) f(v)$. It is a well-known consequence of linear programming duality that the f-fractional chromatic number is equal to the maximum f-weight of a fractional clique [17].

The *2-universal 2-dir graph with m rows* is the horizontally trivial triangle-free 2-dir graph with rows $\ell_1, \ldots, \ell_m$ in order according to the y-coordinate and with m vertical lines, where the first vertical line contains a segment intersecting all rows and for $i \in \{2, \ldots, m\}$, the i-th vertical line contains two intersecting vertical segments, one of them intersecting $\ell_1, \ldots, \ell_{i-1}$ and the other one intersecting $\ell_i, \ldots, \ell_m$.

Lemma 5. *Let $m \geq 1$ be an integer, let $\varepsilon > 0$ be a real number, let G be the 2-universal 2-dir graph with m rows, and let $f : V(G) \to \mathbb{R}^+$ be defined by setting $f(v) = 1$ for each vertical segment v and $f(v) = 2 - \varepsilon$ for each horizontal segment v. Then G has f-fractional chromatic number at least $4 - \frac{1}{m} - \varepsilon$.*

Proof. Let us define $\gamma(v) = \frac{1}{m}$ for every $v \in V(G)$. We claim that γ is a fractional clique. Indeed, consider any independent set I. If I does not contain any horizontal segment, then I contains at most one vertical segment from each vertical line, and thus $|I| \leq m$. Otherwise, let j_1 be the minimum index such that I contains the horizontal segment from row ℓ_{j_1} and let j_2 be the maximum such such index; then I contains at most $j_2 - j_1 + 1$ horizontal segments. Moreover, I cannot contain the vertical segment from the first vertical line or from the i-th vertical line for any $i \in \{j_1 + 1, \ldots, j_2\}$, and thus $|I| \leq (j_2 - j_1 + 1) + m - (j_2 - j_1 + 1) = m$. Therefore, $\sum_{v \in I} \gamma(v) \leq \frac{1}{m}|I| \leq 1$.

The f-weight of γ is $\frac{1}{m}(m \cdot (2 - \varepsilon) + 2m - 1) = 4 - \frac{1}{m} - \varepsilon$, establishing the desired lower bound on the f-fractional chromatic number of G. □

Since the 2-universal 2-dir graph is horizontally trivial, we can combine Lemmas 4 and 5 to obtain the lower bound from Theorem 1.

Corollary 6. *For all integers* $r, m \geq 1$, $\chi_f(\mathcal{D}_{rm,2}) \geq 4 - \frac{1}{m} - \frac{1}{2^{r-1}}$.

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