Proceedings of the 12th European Conference on Combinatorics, Graph Theory and Applications
EUROCOMB'23
Prague, August 28 - September 1, 2023

Exact antichain saturation numbers via a generalisation of a result of Lehman-Ron

(Extended abstract)

Paul Bastide * Carla Groenland † Hugo Jacob ‡ Tom Johnston §

Abstract

For given positive integers k and n, a family $\mathcal{F}$ of subsets of $\{1, \ldots, n\}$ is k-antichain saturated if it does not contain an antichain of size k, but adding any set to $\mathcal{F}$ creates an antichain of size k. We use $\text{sat}^*(n, k)$ to denote the smallest size of such a family. For all k and sufficiently large n, we determine the exact value of $\text{sat}^*(n, k)$. Our result implies that $\text{sat}^*(n, k) = n(k-1) - \Theta(k \log k)$, which confirms several conjectures on antichain saturation. Previously, exact values for $\text{sat}^*(n, k)$ were only known for k up to 6.

We also prove a strengthening of a result of Lehman-Ron which may be of independent interest. We show that given m disjoint chains in the Boolean lattice, we can create m disjoint skipless chains that cover the same elements (where we call a chain skipless if any two consecutive elements differ in size by exactly one).

The complete version of the paper can be found here [4].

DOI: `https://doi.org/10.5817/CZ.MUNI.EUROCOMB23-017`

*LaBRI - University of Bordeaux, `paul.bastide@ens-rennes.fr`

†Utrecht University, `c.e.groenland@uu.nl`, This project has received funding from the European Union's Horizon 2020 research and innovation programme under the ERC grant CRACKNP (number 853234) and the Marie Skłodowska-Curie grant GRAPHCOSY (number 101063180). Views and opinions expressed are however those of the author(s) only.

‡ENS Paris-Saclay, `hjacob@ens-paris-saclay.fr`

§University of Bristol and Heilbronn Institute for Mathematical Research, `tom.johnston@bristol.ac.uk`

Many powerful results have been proved over the years concerning the structure of chains and antichains in the Boolean lattice, e.g. [10, 14, 20, 21, 22]. For example, it is well-known that the Boolean lattice admits a symmetric chain decomposition [1, 9], and in fact these chains may be taken to be *skipless* (or *saturated*): every chain $C_1 \subsetneq \cdots \subsetneq C_r \subseteq [n] = \{1, \ldots, n\}$ has the property that $|C_{i+1}| = |C_i| + 1$ for all $i \in [r-1]$. Skipless chains have also been studied in other contexts such as in [3, 6, 16].

Given sets $X_1, \ldots, X_m$ from layer r and sets $Y_1, \ldots, Y_m$ from layer s such that $X_i \subseteq Y_i$, it need not be possible to find disjoint skipless chains $C^1, \ldots, C^m$ linking X_1 to Y_1, X_2 to Y_2 etc. However, it was shown by Lehman and Ron [15] in 2001 that there always exist m disjoint skipless chains that *cover* the sets $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_m$.

Theorem 1 (Lehman-Ron [15])**.** *Let integers $1 \leq s < r \leq n$ and subsets $X_1, \ldots, X_m, Y_1, \ldots, Y_m \subseteq [n]$ be given with $|X_i| = s$, $|Y_i| = r$ and $X_i \subseteq Y_i$ for all $i \in [m]$. Then there exist m disjoint skipless chains that cover $\{X_1, \ldots, X_m, Y_1, \ldots, Y_m\}$.*

It is natural to ask if a stronger statement holds. For example, what happens if we allow the sets to come from different layers, or ask that the chains go via some elements from layers between layer r and layer s? Is it possible to cover any m disjoint chains with m disjoint skipless chains, or can we force the use of an additional chain? We show that m chains always suffice. For a family $\mathcal{F}$, we say that $\mathcal{F}$ admits a chain decomposition into m chains if there exits m disjoint chains $C_1, \ldots, C_m$ that covers $\mathcal{F}$.

Theorem 2. *Suppose that $\mathcal{F} \subseteq 2^{[n]}$ admits a chain decomposition into m chains. Then there exist disjoint skipless chains $C^1, \ldots, C^m$ such that $\mathcal{F} \subseteq \bigcup_{i=1}^m C^i$.*

Proof overview. The core of the proof, despite being slightly more technical, follows a method similar to the one used in [15]. It uses multiple inductive arguments to reduce the problem to a well-structured instance. From there it is possible to uses Menger's theorem [18] to deduce connectivity properties.

The building blocks for our induction can are as follows (see Fig. 1 for an example):

Claim 3. *Let $s \leq r \leq n$ be integers. Let $C^1, \ldots, C^m$ be disjoint chains, such that for all $i \in [m-1]$, the chain C^i starts in layer s and ends in layer r. Suppose that C^m starts in $A \in \binom{[n]}{\leq s}$ and ends in $B \in \binom{[n]}{r}$. Then there exist m disjoint chains $D^1, \ldots, D^m$ with the following three properties.*

1. *For $i \in [m-1]$, the chain D^i starts in the sth layer, ends in the rth layer and is skipless.*
2. *The chain D^m starts at A and intersects the ith layer for all $i \in [s+1, r]$.*
3. *The chains $D^1, \ldots, D^m$ cover the elements in $C^1, \ldots, C^m$.*

□

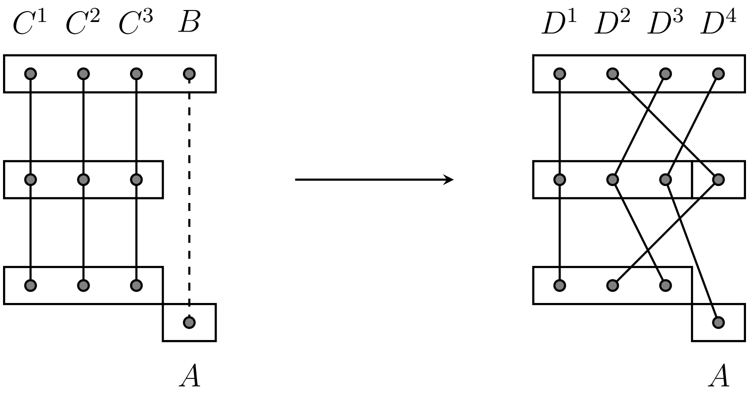

Figure 1: Representation of Claim 3 (case $r = s + 2$ and $m = 3$).

Theorem 2 was already known in the special case that the union $\mathcal{F}$ of the chains we wish to cover is a convex set system (i.e. if $X, Y \in \mathcal{F}$ and $X \subseteq Z \subseteq Y$, then $Z \in \mathcal{F}$) [6]. In this case, the chains can be taken to partition $\mathcal{F}$ as any additional sets must be at the ends of the chains.

Although we believe Theorem 2 to be of interest in its own right, our initial motivation came from the area of induced poset saturation where we use Theorem 2 to easily settle various conjectures concerning the asymptotics of antichain saturation numbers. With more work, we are in fact able to go well beyond the conjectures and pinpoint the exact values.

For given positive integers k and n, a family $\mathcal{F}$ of subsets of $[n]$ is *k-antichain saturated* if it does not contain an antichain of size k, but for all $X \subseteq [n]$ with $X \notin \mathcal{F}$, the family $\mathcal{F} \cup \{X\}$ does contain an antichain of size k. We denote the size of the smallest such family by $\mathrm{sat}^*(n, k)$.

In the literature, this is also sometimes denoted $\mathrm{sat}^*(n, \mathcal{A}_k)$, where $\mathcal{A}_k$ is the poset consisting of k incomparable elements. This is called an *induced* saturation number: it is the size of the smallest set system which is saturated in terms of not containing $\mathcal{A}_k$ as an *induced* subposet. Such saturation numbers for the Boolean lattice were introduced by Gerbner, Keszegh, Lemons, Palmer, Pálvölgyi and Patkós [8] and have been investigated for a variety of posets, for example for the butterfly [11], the diamond [12] and the chain [19]. We refer to [13] for a nice overview.

Ferrara, Kay, Kramer, Martin, Reiniger, Smith and Sullivan [7] were the first to study the particular case of the antichain and made the following conjecture.

Conjecture 1 ([7]). *For $k \geq 3$, $\mathrm{sat}^*(n, k) \sim (k-1)n$ as $n \to \infty$.*

The upper bound is easy to see: for all $i \in [n]$, a k-antichain saturated family can contain at most $k - 1$ subsets of size i since two subsets of the same size are incomparable. Moreover, a k-antichain saturated family must always exist since we can start with the empty family and greedily add subsets until it is no longer possible to do so without creating an antichain of size k.

Martin, Smith and Walker [17] proved the lower bound

$$\mathrm{sat}^*(n,k) \geq \left(1 - \frac{1}{\log_2(k-1)}\right) \frac{(k-1)n}{\log_2(k-1)}$$

for $k \geq 4$ and n sufficiently large. The exact values for $k = 2, 3$ and 4 were shown to be $n+1$, $2n$ and $3n-1$ respectively in [7], the exact values for $k = 5$ and $k = 6$ were recently determined to be $4n-2$ and $5n-5$ respectively by Đanković and Ivan [2]. They also strengthened Conjecture 1 as follows, and proposed two weaker conjectures implied by this conjecture.

Conjecture 2 ([2]). $\mathrm{sat}^*(n,k) = n(k-1) - O_k(1)$.

We show later how all the conjectures mentioned above are easily derived from Theorem 2. In particular, we will show the following corollary.

Corollary 4. *There exist constants $c_1, c_2 > 0$ such that for all $k \geq 4$ and n sufficiently large,*

$$n(k-1) - c_1 k \log k \leq \mathrm{sat}^*(n,k) \leq n(k-1) - c_2 k \log k.$$

In general, obtaining exact saturation numbers is a notoriously difficult problem, and for the antichain exact numbers were only known for k up to 6. Our main result determines the exact value of $\mathrm{sat}^*(n,k)$ for all values of k and n where n is large enough relative to k . We note that n need not be excessively large compared to k and it certainly suffices to assume $n \geq 6 \log k + 1$ for example. Determining the exact values is considerably more involved than just determining the asymptotics, and we require some more definitions just to state the value of the numbers.

Given a natural number k, let ℓ be the smallest integer j such that $\binom{j}{\lfloor j/2 \rfloor} \geq k-1$. Note that when $n < \ell$, there are no antichains of size k in $2^{[n]}$ and $\mathcal{F}$ must contain every set (i.e. $\mathrm{sat}^*(n,k) = 2^n$).

Let $\mathcal{C}(m,t)$ denote the initial segment of layer t of size m when the sets are in colexicographic order. For a family of sets $\mathcal{A}$ from the same layer, let $\nu(\mathcal{A})$ be the size of the maximum matching from $\mathcal{A}$ to its shadow $\partial\mathcal{A}$, and recursively define $c_0, c_1, \ldots, c_{\lfloor \ell/2 \rfloor}$ as follows. Let $c_{\lfloor \ell/2 \rfloor} = k-1$. For $0 \leq t < \lfloor \ell/2 \rfloor$, let $c_t = \nu\left(\mathcal{C}(c_{t+1}, t+1)\right)$.

Theorem 5. *Let $n, k \geq 4$ be integers and let ℓ and $c_0, \ldots, c_{\lfloor \ell/2 \rfloor}$ be as defined above. If $n < \ell$, then $\mathrm{sat}^*(n,k) = 2^n$. If $n \geq \ell$, then*

$$\mathrm{sat}^*(n,k) \geq 2 \sum_{t=0}^{\lfloor \ell/2 \rfloor} c_t + (k-1)(n-1-2\lfloor \ell/2 \rfloor).$$

Moreover, equality holds when $n \geq 2\ell + 1$.

Given the form of the bound in Theorem 5, one might be tempted to suggest that the best approach is to take each layer $t \leq \lfloor \ell/2 \rfloor$ to be an initial segment of colex of the appropriate size, but this is not the case in general. While such an example would have the optimal size, it may already contain an antichain of size k. For example, one can check there is an antichain of size 262 in $\mathcal{C}(261,5) \cup \mathcal{C}(219,4)$, and this approach would not work for $k = 262$.

For infinitely many values of k, a matching upper bound to Theorem 5 was already known [7] which works for all $n \geq \ell + 1$. It gives the following corollary.

Corollary 6. *Let ℓ, k, n be integers such that $\binom{\ell}{\lfloor \ell/2 \rfloor} = k - 1$. If $n \leq \ell$ then $\mathrm{sat}^*(n,k) = 2^n$. If $n \geq \ell + 1$, then*

$$\mathrm{sat}^*(n,k) = 2 \sum_{j=0}^{\lfloor \ell/2 \rfloor} \binom{\ell}{j} + (k-1)(n - 1 - 2\lfloor \ell/2 \rfloor).$$

In particular, whenever $k-1$ is a central binomial coefficient (i.e. $k = 3, 4, 7, 11, 21, 36, \dots$) the value of $\mathrm{sat}^*(n,k)$ is determined for all n.

We now explain how Corollary 4 follows from Theorem 2. The upper bound was already known, and we prove a lower bound of $\mathrm{sat}^*(n,k) \geq (n + 1 - 2\ell)(k-1)$ for n sufficiently large. (Recall that ℓ is the smallest j such that $\binom{j}{\lfloor j/2 \rfloor} \geq k - 1$, so $\ell = \Theta(\log k)$.)

By Dilworth's theorem [5], having a chain decomposition of size at most $k-1$ is equivalent to not containing any antichain of size k. Suppose that $\mathcal{F} \subseteq 2^{[n]}$ is k-antichain saturated and so admits a decomposition into $k-1$ chains. By Theorem 2, there are $k-1$ disjoint skipless chains $C^1, \dots, C^{k-1}$ that cover the elements of $\mathcal{F}$; since $\mathcal{F}$ is saturated, this must form a chain decomposition of $\mathcal{F}$. It suffices to show that every chain must contain a set of size at most ℓ and a set of size at least $n - \ell$. Suppose the smallest element X of some chain C^i has size $|X| > \ell$, then all subsets Y of X must be present in $\mathcal{F}$ since otherwise we may extend C^i to include Y (and that would mean that $\mathcal{F} \cup \{Y\}$ can also be covered by $k-1$ chains, contradicting the fact that $\mathcal{F}$ is k-antichain saturated). There are at least $k-1$ subsets of X of size $\lfloor \ell/2 \rfloor$, and these cannot all be covered by the other $k-2$ chains. Since each chain contains an element of size at most ℓ and one of size at least $n-\ell$, the bound follows immediately from the fact that the chains are skipless.

In order to prove the exact lower bound of Theorem 5, it is needed to examine what happens on layers $1, \dots, \ell$. This is considerably more delicate and for this we use an auxiliary result concerning the matching number of the colex order, explained in details in the complete version of the paper [4]. There, we also give an explicit construction of a k-antichain saturated system $\mathcal{F}$ which matches our lower bound on each layer provided n is sufficiently large. This construction was already known for the special case $k - 1 = \binom{\ell}{\lfloor \ell/2 \rfloor}$, and we apply it recursively for other values of k. The recursion requires special care and depends on a particular way of writing $k-1$ as a sum of binomial coefficients. This notation can be used to write exact values for the matching numbers c_t from Theorem 5.

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