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Nonrepetitive colorings of $\mathbb{R}^d$

(Extended abstract)

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Abstract

The results of Thue state that there exists an infinite sequence over 3 symbols without 2 identical adjacent blocks, which we call a 2-nonrepetitive sequence, and also that there exists an infinite sequence over 2 symbols without 3 identical adjacent blocks, which is a 3-nonrepetitive sequence. An r*-repetition* is defined as a sequence of symbols consisting of r identical adjacent blocks, and a sequence is said to be r-*nonrepetitive* if none of its subsequences are r-repetitions. Here, we study colorings of Euclidean spaces related to the work of Thue. A coloring of $\mathbb{R}^d$ is said to be r-*nonrepetitive* of no sequence of colors derived from a set of collinear points at distance 1 is an r-repetition. In this case, the coloring is said to *avoid* r-repetitions. It was proved in [9] that there exists a coloring of the plane that avoids 2-repetitions using 18 colors, and conversely, it was proved in [3] that there exists a coloring of the plane that avoids 43-repetitions using only 2 colors. We specifically study r-nonrepetitive colorings for fixed number of colors : for a fixed number of colors k and dimension d, the aim is to determine the minimum multiplicity of repetition r such that there exists an r-nonrepetitive coloring of $\mathbb{R}^d$ using k colors.

We prove that the plane, $\mathbb{R}^2$, admits a 2- and a 3-coloring avoiding 33- and 18-repetitions, respectively.

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1 Introduction

The Hadwiger-Nelson problem asks for the minimum number of colors required to color the Euclidean plane such that any two points at distance 1 are colored differently. This is called the chromatic number of the plane, and is denoted as $\chi(\mathbb{R}^2)$. The answer to this problem is unknown, but it was proved that $5 \leq \chi(\mathbb{R}^2) \leq 7$ [1, 2, 7]. We study colorings of Euclidean spaces that are connected to the Hadwiger-Nelson problem and where the goal is to avoid specific patterns on straight lines.

An r-*repetition* is a finite sequence of symbols consisting of r identical blocks, where a *block* is a subsequence of consecutive terms. A sequence is r-nonrepetitive if none of its subsequences of consecutive terms are r-repetitions. For instance, the word *hotshots* is a 2-repetition and the word *minimize* is 2-nonrepetitive. The results of Thue state that there exists an infinite 2-nonrepetitive sequence over 3 symbols and an infinite 3-nonrepetitive sequence over 2 symbols. We study the Euclidean variant of Thue sequences introduced by Grytczuk *et al.* [5]. A *straight path* is defined as a sequence of collinear points of $\mathbb{R}^d$, where consecutive points are at distance 1. A coloring of $\mathbb{R}^d$ is r-*nonrepetitive* if for each straight path in $\mathbb{R}^d$, the sequence of the colors of its points is r-nonrepetitive. For fixed integers d and r, the aim is to find the minimum number of colors for which there exists an r-nonrepetitive coloring of $\mathbb{R}^d$. Let $\pi_r(\mathbb{R}^d)$ denote that number.

One easily deduces from Thue's result that $\pi_2(\mathbb{R}) = 3$ and $\pi_3(\mathbb{R}) = 2$. The problem is more difficult for higher dimensions. Colorings of Euclidean spaces that avoid 2-repetitions are called *square-free* colorings. It was proven in [9] that there exists a square-free coloring of the plane that uses 18 colors, which means that $\pi_2(\mathbb{R}^2) \leq 18$. The problem of determining $\pi_2(\mathbb{R}^2)$ is connected to the Hadwiger-Nelson problem in the following way. If a coloring of the plane is 2-nonrepetitive, then 2 points at distance 1 must be colored differently, so at least $\chi(\mathbb{R}^2)$ colors are required. Therefore $5 \leq \chi(\mathbb{R}^2) \leq \pi_2(\mathbb{R}^2) \leq 18$.

Dębski *et al.* studied r-nonrepetitive colorings for larger values of r [3]. More specifically, they gave a proof that for any $d \in \mathbb{N}$, there exists $r = r(d)$ such that $\pi_r(\mathbb{R}^d) = 2$. In other words, for large enough values of r, the problem can be solved with the least possible number of colors. In particular, for $d = 2$, the minimum value of r for which $\pi_r(\mathbb{R}^2) = 2$ is unknown, but the paper provides a proof that $\pi_{43}(\mathbb{R}^2) = 2$ and $\pi_{24}(\mathbb{R}^2) \leq 3$. For smaller values of r, it is known that $\pi_6(\mathbb{R}^2) \leq 4$ and $\pi_3(\mathbb{R}^2) \leq 9$ [4, 9].

We prove that there exists a 33-nonrepetitive coloring of $\mathbb{R}^2$ with 2 colors, that is, $\pi_{33}(\mathbb{R}^2) = 2$. We also prove that $\pi_{18}(\mathbb{R}^2) \leq 3$. Our improvements rely on two main ingredients. First, we provide a better bound on the number of *pathable sequences of hypercubes*. This quantity already played a crucial role in the proof from [3]. Secondly, the proof from [3] uses the Lovász Local Lemma, which we replace with a counting method that yields slightly better bounds in this setting. This argument was first used for nonrepetitive colorings of graphs [6] and was later presented in the more general context of hypergraph coloring [8].

2 Pathable sequences

A standard technique in problems related to colorings of Euclidean spaces is to define a regular tiling of that space and assign the same color to all the points of each tile. The proof of the result from [3] uses a partition of $\mathbb{R}^d$ into hypercubes of diameter 1. We will also use this partition. More precisely, each hypercube is a set of the form $\{(x_1, ..., x_d) \in \mathbb{R}^d : \forall j \in [\![1, d]\!], i_j \leq x_j\sqrt{d} < i_j + 1\}$, with $(i_1, ..., i_d) \in \mathbb{Z}^d$. This way, any two points at distance 1 are always in different hypercubes. Let $\mathcal{H}$ denote the set of hypercubes from this partition.

We call a sequence $(\alpha_0, ..., \alpha_{\ell-1})$ of hypercubes ℓ-pathable if there exists a straight path $(q_0, ..., q_{\ell-1})$ in $\mathbb{R}^d$ with $q_i \in \alpha_i$ for each i (See Figure 1). For a fixed cube H, $D_d(\ell)$ is defined as the number of ℓ-pathable sequences in $\mathbb{R}^d$ containing H (each pair $(\alpha_0, ..., \alpha_{\ell-1})$ and $(\alpha_{\ell-1}, ..., \alpha_0)$ is counted as a single sequence).

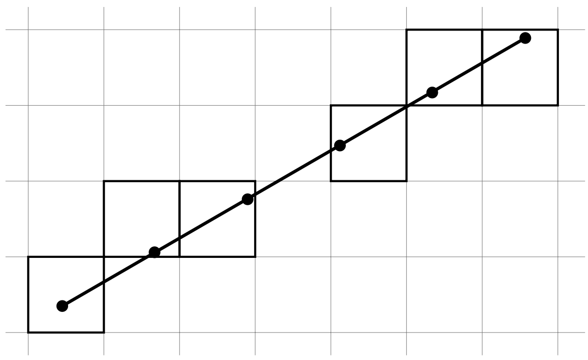

Figure 1: A 6-pathable sequence in $\mathbb{R}^2$.

It is know that $D_d(l) = O(l^{3d})$ [3]. We improve this upper bound for $d = 2$.

Lemma 1. *The number of ℓ-pathable sequences in $\mathbb{R}^2$ is bounded as follows,*

$$D_2(\ell) \leq \frac{2\sqrt{2}}{3}\ell^5 + (2 - \frac{2\sqrt{2}}{3})\ell^3 - 2\ell^2 \,.$$

3 Calculations with the counting argument

In this section, we provide a condition similar to [3, Lemma 2.3]. It provides a condition on r and the number of colors k that ensures that there exists an r-nonrepetitive coloring of $\mathbb{R}^d$ using k colors. However, the condition of Lemma 3, can be proven to be weaker, that is, whenever the condition of [3, Lemma 2.3] holds then our lemma automatically holds with $\beta = k2^{-1/(r-1)}$. In practice, this leads to a slightly better bound for our results.

In the proof of Lemma 2, we will consider an arbitrary subset S of $\mathbb{R}^d$ consisting of finitely many hypercubes from the partition. This method directly shows that there exist exponentially many valid hypercube colorings, with respect to the number of hypercubes in S.

Lemma 2. *Let r, k and d be integers. For every set S of hypercubes, let $\mathcal{C}(S)$ be the set of r-nonrepetitive hypercube colorings of S with k colors.*

If there exists $\beta > 1$ such that

$$k \geq \beta + \sum_{s=1}^{\infty} D_d(rs) \times \beta^{1-(r-1)s},$$

then for every set S of n hypercubes of the partition of $\mathbb{R}^d$ and for every hypercube $H \in S$,

$$|\mathcal{C}(S)| \geq \beta |\mathcal{C}(S-H)|.$$

Remark that $\beta > 1$ and that according to Corollary 2.5 from [3], $D_d(rs) = O((rs)^{3d})$, so the sum in this Lemma is always well-defined.

Proof. We proceed by induction on $n = |S|$. This is true for $n = 1$ because $S - H = \emptyset$. Fix $n \geq 2$ and assume that the result holds for every $i < n$. Let S be a set of n hypercubes and H a hypercube of S. Our induction hypothesis implies that for all $R \subseteq S - H$,

$$\mathcal{C}(S-H-R) \leq \frac{\mathcal{C}(S-H)}{\beta^{|R|}}. \tag{1}$$

Let F be the set of colorings of S that are r-nonrepetitive on $S-H$ but for which there is an r-repetition on S. Then

$$|\mathcal{C}(S)| = k|\mathcal{C}(S-H)| - |F|. \tag{2}$$

Let $s \in \mathbb{N}^*$ and $\alpha = (\alpha_1, ..., \alpha_{rs})$ be a pathable sequence such that $H = \alpha_i$, for some $i \in \{1, \ldots, rs\}$. We define F_α as the subset of F for which there is an r-repetition of length rs on that sequence. Without loss of generality, we assume that $i \geq s+1$. We consider a coloring $\phi \in F_\alpha$. By definition of F, the sequence of colors on α is an r-repetition, and the restriction of ϕ to $S - (\alpha_{s+1}, ..., \alpha_{rs})$ is r-nonrepetitive because $H \in \{\alpha_{s+1}, ..., \alpha_{rs}\}$. Therefore, ϕ is uniquely determined by its restriction to $S - \{\alpha_{s+1}, ..., \alpha_{rs}\}$ and $|F_\alpha| \leq |\mathcal{C}(S - \{\alpha_{s+1}, ..., \alpha_{rs}\}|$. By equation (1), this implies,

$$|F_\alpha| \leq \frac{1}{\beta^{(r-1)s-1}} |\mathcal{C}(S-H)|.$$

Let F_{rs} be the subset of F for which there is an r-repetition of length rs. Recall that $D_d(rs)$ is the number of pathable sequences of length rs containing H. Then,

$$|F_{rs}| \leq D_d(rs) \frac{1}{\beta^{(r-1)s-1}} |\mathcal{C}(S-H)|.$$

Now, by summing over all s, and by using our main hypothesis

$$|F| = \left| \bigcup_{s=1}^{\infty} F_{rs} \right| \leq \sum_{s=1}^{\infty} |F_{rs}| \leq \sum_{s=1}^{\infty} D_d(rs) \frac{1}{\beta^{(r-1)s-1}} |\mathcal{C}(S-H)| \leq |\mathcal{C}(S-H)|(k-\beta).$$

Using this bound inside equation (2),

$$|\mathcal{C}(S)| = k|\mathcal{C}(S-H)| - |F| \geq \beta|\mathcal{C}(S-H)|$$

which concludes our induction. □

For each subset S of $\mathbb{R}^d$ consisting of n hypercubes, $|\mathcal{C}(S)| \geq \beta^{n-1}k$. This means that any finite arbitrary subset of hypercubes of the partition of $\mathbb{R}^d$ can be r-nonrepetitively colored. By compacity (e.g., see the proof of Lemma 2.3 from [3]) there exists an r-nonrepetitive coloring of $\mathbb{R}^d$.

Lemma 3. *For every integers r, k and d, if there exists $\beta > 1$ such that*

$$k \geq \beta + \sum_{s=1}^{\infty} D_d(rs) \times \beta^{1-(r-1)s}$$

then $\pi_r(\mathbb{R}^d) \leq k$.

4 Proof of the main results and conclusion

We can now use the bound from Lemma 1 to verify the conditions of Lemma 3 for well-chosen values of r, β and k. In particular, one can verify that the condition of Lemma 3 holds for $r = 33$, $\beta = 19/10$ and $k = 2$ which implies the following result.

Theorem 4. *There exists a 2-coloring of the plane avoiding 33-repetitions.*

Let $r(d)$ denote the least positive integer such that $\pi_{r(d)}(\mathbb{R}^d) = 2$. We proved that $r(2) \leq 33$, which improves the bound $r(2) \leq 43$ proved in [3]. However, this result probably isn't optimal, since the best known lower bound is $r(2) \geq 3$ which is a consequence of the results of Thue. This means that $r(2)$ lies between 3 and 33. In fact, it is conjectured in [3] that $r(2) = 4$.

Similarly, one can verify that the condition of Lemma 3 holds for $r = 18$, $\beta = 8/3$ and $k = 3$ which implies the following result.

Theorem 5. *There exists a 3-coloring of the plane avoiding 18-repetitions.*

Again the value 18 is an improvement from 24 but is probably still not optimal.

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