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Isoperimetric stability in lattices

(Extended abstract)

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Abstract

We obtain isoperimetric stability theorems for general Cayley digraphs on $\mathbb{Z}^d$. For any fixed B that generates $\mathbb{Z}^d$ over $\mathbb{Z}$, we characterise the approximate structure of large sets A that are approximately isoperimetric in the Cayley digraph of B: we show that A must be close to a set of the form $kZ \cap \mathbb{Z}^d$, where for the vertex boundary Z is the conical hull of B, and for the edge boundary Z is the zonotope generated by B.

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1 Introduction

An important theme at the interface of Geometry, Analysis and Combinatorics is understanding the structure of approximate minimisers to isoperimetric problems. These problems take the form of minimising surface area of sets with a fixed volume, for various meanings of 'area' and 'volume'. The usual meanings give the Euclidean Isoperimetric Problem considered since the ancient Greek mathematicians, where balls are the measurable subsets of $\mathbb{R}^d$ with a given volume which minimize the surface area. There is a

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large literature on its stability, i.e. understanding the structure of approximate minimisers, culminating in the sharp quantitative isoperimetric inequality of Fusco, Maggi and Pratelli [8].

In the discrete setting, isoperimetric problems form a broad area that is widely studied within Combinatorics (see the surveys [2, 14]) and as part of the Concentration of Measure phenomenon (see [15, 26]). Certain particular settings have been intensively studied due to their applications; for example, there has been considerable recent progress (see [12, 11, 13, 23]) on isoperimetric stability in the discrete cube $\{0,1\}^n$, which is intimately connected to the Analysis of Boolean Functions (see [20]) and the Kahn–Kalai Conjecture (see [10]) on thresholds for monotone properties, which has recently been solved [7, 21]. This paper concerns the setting of integer lattices, which is widely studied in Additive Combinatorics, where the Polynomial Freiman–Ruzsa Conjecture (see [9]) predicts the structure of sets with small doubling.

For an isoperimetric problem on a digraph (directed graph) G, we measure the 'volume' of $A \subseteq V(G)$ by its size $|A|$, and its 'surface area' either by the *edge boundary* $\partial_{e,G}(A)$, which is the number of edges $\overrightarrow{xy} \in E(G)$ with $x \in A$ and $y \in V(G) \setminus A$, or by the *vertex boundary* $\partial_{v,G}(A)$, which is the number of vertices $y \in V(G) \setminus A$ such that $\overrightarrow{xy} \in E(G)$ for some $x \in A$. Here we consider Cayley digraphs: given a generating set B of $\mathbb{Z}^d$, we write G_B for the digraph on $\mathbb{Z}^d$ with edges $E(G_B) = \{\overrightarrow{uv} : v - u \in B\}$.

It is an open problem to determine the minimum possible value of $\partial_{v,G_B}(A)$ or $\partial_{e,G_B}(A)$ for $A \subseteq \mathbb{Z}^d$ of given size, let alone any structural properties of (approximate) minimisers; exact results are only known for a few instances of B (see [3, 4, 27, 24]). It is therefore natural to seek asymptotics. For ease of reference we collect here our notation for the various sets involved in stating the following results.

- $C(B) \subseteq \mathbb{R}^d$ — The *conical hull* $C(B)$ of B is the convex hull of $B \cup \{0\}$.
- $B_n \subseteq \mathbb{Z}^d$ — The sets $kC(B) \cap \mathbb{Z}^d$ are increasing as a function of $k > 0$. Write B_n for the smallest of these sets with at least n elements.
- $[B] \subseteq \mathbb{Z}^d$ — Write $[B] = \left\{ \sum_{b \in B'} b : B' \subseteq B \right\}$ for the set of all sums of subsets of B. Thus $|[B]| \leq 2^{|B|}$, where the bound is strict if multiple subsets of B have equal sums.
- $Z(B) \subseteq \mathbb{R}^d$ — The *zonotope* generated by B is $\left\{ \sum_{b \in B} x_b b : x \in [0,1]^B \right\}$. Equivalently, $Z(B)$ is the convex (or conical, as $[B]$ contains 0) hull of $[B]$.

For $A \subseteq \mathbb{Z}^d$ of size $n \to \infty$, Ruzsa [25] showed that the minimum value of the vertex boundary $\partial_{v,G_B}(A)$ is asymptotic to that achieved by a set of the form $kC(B) \cap \mathbb{Z}^d$. A corresponding result for the edge boundary was obtained in [1]: the minimum value of $\partial_{e,G_B}(A)$ is asymptotic to that achieved by a set of the form $kZ(B) \cap \mathbb{Z}^d$.

We will prove stability versions of both these results, describing the approximate structure of asymptotic minimisers for both the vertex and edge isoperimetric problems in G_B. We use μ to denote Lebesgue measure.

Theorem 1.1. *Let $d \geq 2$. For every generating set B of $\mathbb{Z}^d$, there is a $K \in \mathbb{N}$ such that whenever*

- $A \subseteq \mathbb{Z}^d$ *with* $|A| = n \geq K$,
- $Kn^{-1/2d} < \varepsilon < K^{-1}$ *and*
- $\partial_{v,G_B}(A) \leq d\mu(C(B))^{1/d} n^{1-1/d}(1+\varepsilon)$,

there is a $v \in \mathbb{Z}^d$ *with* $|A \triangle (v + B_n)| < Kn\sqrt{\varepsilon}$.

Theorem 1.2. *Let $d \geq 2$. For every generating set B of $\mathbb{Z}^d$ and $\delta > 0$, there are $K \in \mathbb{N}$ and $\epsilon > 0$ such that whenever*

- $A \subseteq \mathbb{Z}^d$ *with* $|A| = n \geq K$ *and*
- $\partial_{e,G_B}(A) \leq d\mu(Z(B))^{1/d} n^{1-1/d}(1+\varepsilon)$,

there is a $v \in \mathbb{Z}^d$ *with* $|A \triangle (v + [B]_n)| < \delta n$.

The square root dependence in Theorem 1.1 is tight, as may be seen from an example where B consists of the corners of a cube and A is an appropriate cuboid.

Besides drawing on the methods of [25] (particularly Plünnecke's inequality for sumsets) and [1] (a probabilistic reduction to [25]), the most significant new contribution of our paper is a technique for transforming discrete problems to a continuous setting where one can apply results from Geometric Measure Theory. We will employ the sharp estimate on asymmetric index in terms of anisotropic perimeter with respect to any convex set K due to Figalli, Maggi and Pratelli [6] (building on the case when K is a ball, established in [8]).

2 Proof strategy

This section contains an overview of the proof of our tight quantitative stability result for the vertex isoperimetric inequality in general Cayley digraphs. Using ideas from [1] one can deduce from this also a stability result for the edge isoperimetric inequality.

We start with a summary of Ruzsa's approach in [25], during which we record some key lemmas on sumsets and fundamental domains of lattices that we will also use in our proof.

2.1 Ruzsa's approach

The sumset of $A, B \in \mathbb{Z}^d$ is defined by $A + B := \{a + b : a \in A, b \in B\}$. The vertex isoperimetric problem in the Cayley digraph G_B is equivalent to finding the minimum of $|A+B|$ over all sets A of given size. The following result of Ruzsa [25, Theorem 2] implies an asymptotic for this minimum.

Theorem 2.1. *Let B be a generating set of $\mathbb{Z}^d$ with $d \geq 2$. Then for any $A \subseteq \mathbb{Z}^d$ with $|A| = n$ large we have $|A + B| \geq d\mu(C(B))^{1/d} n^{1-1/d}\big(1 - O(n^{-1/2d})\big)$.*

Ruzsa aims to deduce this inequality from the Brunn–Minkowski inequality (in the form due to Lusternik [16]) $\mu(U+V)^{1/d} \geq \mu(U)^{1/d} + \mu(V)^{1/d}$, which is tight when U and V are closed, convex and homothetic (that is, agree up to scaling and translation).

Passing from a discrete inequality to a continuous one can be achieved by adding a *fundamental set* Q to each side; that is, a measurable Q such that any $x \in \mathbb{R}^d$ has a unique representation as $x = z + q$ with $z \in \mathbb{Z}^d$ and $q \in Q$. This ensures that $\mu(X+Q) = |X|$ for any $X \subseteq \mathbb{Z}^d$. One example of a fundamental set is the half-open unit cube $[0,1)^d$, but we will prefer a fundamental set tailored to B rather than to the standard coordinate axes.

Typically $B+Q$ will be far from convex, so a naive application of Brunn–Minkowski gives poor results. Ruzsa smooths out B by using a version of Plünnecke's inequality [22] to replace B by its sumset. We write $\Sigma_k(A)$ for the k-fold sumset of A rather than the commonly used kA, which in this paper denotes the dilate of A by factor k.

Theorem 2.2 (see [25, Statement 6.2])**.** *Let $k \in \mathbb{N}$ and $A, B \subseteq \mathbb{Z}^d$ with $|A| = n$ and $|A+B| = \alpha n$. Then there is a non-empty subset $A' \subseteq A$ with $|A' + \Sigma_k(B)| \leq \alpha^k |A'|$.*

To return to a bound on to discrete sets Ruzsa uses the following lemma. By *nice* we mean that a set is a finite union of bounded convex polytopes.

Lemma 2.3 ([25, Lemma 11.2])**.** *Let B be a generating set of $\mathbb{Z}^d$ with $d \geq 2$ and $0 \in B$. Then there are $p \in \mathbb{N}$, $z \in \mathbb{Z}^d$ and a nice fundamental set $Q \subseteq Z(B)$ such that $kC(B) + Q + z \subseteq \Sigma_{k+p}(B) + Q$ for any $k \in \mathbb{N}$.*

The fact that Q may be chosen to be nice and such that $Q \subseteq Z(B)$ is not stated in [25], but it can be read out of the proof. With a little care Q can be taken to be a parallelepiped, but we make no use of this observation.

Chaining together the inequalities in this section and optimising over k proves Theorem 2.1. A similar process, taking notice of the stability of our application of the Brunn–Minkowski inequality, will prove Theorem 1.1.

2.2 Some Geometric Measure Theory

The next element of our proof incorporates a recent quantitative isoperimetric stability result of Figalli, Maggi and Pratelli [6]. We adopt simplified definitions that suffice for sets that are nice, as defined in the previous subsection; see [17, 18] for the general setting of sets of finite perimeter.

For a closed convex polytope $K \subseteq \mathbb{R}^d$ and a union E of disjoint (possibly non-convex) closed polytopes, the perimeter of E with respect to K is given by

$$\mathrm{Per}_K(E) = \lim_{\varepsilon \to 0^+} \frac{\mu(E + \varepsilon K) - \mu(E)}{\varepsilon}. \tag{1}$$

In our setting, given a nice set A, for all $r \geq 0$ the measure of $A + rK$ and its closure $\overline{A + rK}$ are the same; that is $\mu(A + rK) = \mu(\overline{A + rK})$. Thus for all $r \geq 0$, (1) gives

$$\mathrm{Per}_K(\overline{A + rK}) = \lim_{\varepsilon \to 0^+} \frac{\mu(A + (r+\varepsilon)K) - \mu(A + rK)}{\varepsilon}. \tag{2}$$

The anisotropic isoperimetric problem was posed in 1901 by Wulff [28], who conjectured that minimisers of Per_K up to null sets are homothetic copies of K, giving $\mathrm{Per}_K(E) \geq d\mu(K)^{1/d}\mu(E)^{1-1/d}$. This was established for sets E with continuous boundary by Dinghas [5] and for general sets E of finite perimeter by Gromov [19]. It is equivalent to non-negativity of the *isoperimetric deficit* $\delta_K(E)$ of E with respect to K, defined by

$$\delta_K(E) := \frac{\mathrm{Per}_K(E)}{d\mu(K)^{1/d}\mu(E)^{1-1/d}} - 1.$$

We quantify the structural similarity between K and E via the *asymmetric index* (also known as Fraenkel asymmetry) of E with respect to K, which is given by

$$\mathcal{A}_K(E) = \inf\left\{\frac{\mu(E \triangle (x_0 + rK))}{\mu(E)} : x_0 \in \mathbb{R}^d \text{ and } r^d\mu(K) = \mu(E)\right\}.$$

Theorem 2.4 ([6, Theorem 1.1])**.** *For any $d \in \mathbb{N}$ there exists $D = D(d)$ such that for any bounded convex open set $K \subseteq \mathbb{R}^d$ and $E \subseteq \mathbb{R}^d$ of finite perimeter we have*

$$\mathcal{A}_K(E) \leq D\sqrt{\delta_K(E)}.$$

2.3 Stability

Given these ingredients, let us indicate briefly how Theorem 1.1 follows.

Given a set A which is close to optimal in terms of Theorem 1.2, using Ruzsa's interpretation of the problem in terms of sumsets, we can apply Lemma 2.2 to find a subset $A' \subseteq A$ which is close to optimal in the lattice generated by $\Sigma_{k+p}(B)$. In particular, this leads to a lower bound on the size of A' in terms of $|A' + \Sigma_{k+p}(B)|$. By taking a continuous approximation of this sumset and applying the Brunn-Minkowski inequality we can conclude that $|A'|$ is approximately $|A|$, and so it suffices to show that A' is structurally close to to an appropriate B_n.

Using Lemma 2.3 we can approximate $\Sigma_{k+p}(B)$ by a homothetic copy of $C(B)$, after thickening by an appropraite fundamental set, and hence relate the boundary in this new lattice to the isoperimetric deficit of A' with respect to $C(B)$. In particular, by Theorem 2.4 we can use this to bound the asymmetric index of A' with respect to $C(B)$, and hence by another discrete approximation, to bound the symmetric difference between A' and some B_n.

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