Proceedings of the 12th European Conference on Combinatorics, Graph Theory and Applications
EUROCOMB'23
Prague, August 28 - September 1, 2023

ON THE SIZES OF t-INTERSECTING k-CHAIN-FREE FAMILIES

(EXTENDED ABSTRACT)

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Abstract

A set system $\mathcal{F}$ is *t-intersecting*, if the size of the intersection of every pair of its elements has size at least t. A set system $\mathcal{F}$ is *k-Sperner*, if it does not contain a chain of length $k+1$.

Our main result is the following: Suppose that k and t are fixed positive integers, where $n+t$ is even and n is large enough. If $\mathcal{F} \subseteq 2^{[n]}$ is a t-intersecting k-Sperner family, then $\mathcal{F}$ has size at most the size of the sum of k layers, of sizes $(n+t)/2, \dots, (n+t)/2+k-1$. This bound is best possible. The case when $n+t$ is odd remains open.

DOI: https://doi.org/10.5817/CZ.MUNI.EUROCOMB23-014

1 Introduction

1.1 Definitions and Notation

For a positive integer n, we write $[n] := \{1, 2, \dots, n\}$ and $2^{[n]}$ for the power set of $[n]$. For a set S, we denote by $\binom{S}{i}$ the family of all i element subsets of S.

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For a family of sets $\mathcal{F} \subseteq 2^{[n]}$, we define $\mathcal{F}_i := \{F \in \mathcal{F} : |F| = i\}$ and $f_i := |\mathcal{F}_i|$. We use Δ_i and ∇_i to denote the i-shadow and i-shade of $\mathcal{F}$, respectively, so that $\Delta_i\mathcal{F} := \{A : |A| = i,\ A \subset F \text{ for some } F \in \mathcal{F}\}$ and $\nabla_i\mathcal{F} := \{A : |A| = i,\ A \supset F \text{ for some } F \in \mathcal{F}\}$. If the subscript i is unspecified, then assuming $\mathcal{F}$ is r-uniform, $\Delta\mathcal{F} = \Delta_{r-1}\mathcal{F}$ and similarly $\nabla\mathcal{F} = \nabla_{r+1}\mathcal{F}$.

Definition 1.1. [k-Sperner family]
A $(k+1)$-*chain* is a collection of $k+1$ sets $A_0, A_1, \ldots, A_k$ such that $A_0 \subset A_1 \subset \ldots \subset A_k$. A family of sets $\mathcal{F} \subseteq 2^{[n]}$ is a *k-Sperner family* if there is no $(k+1)$-chain in $\mathcal{F}$. If $k = 1$, then $\mathcal{F}$ is simply called a *Sperner family* or an *antichain.*

Definition 1.2. [t-intersecting family]
A family of sets $\mathcal{F} \subseteq 2^{[n]}$ is *t-intersecting* if for every pair of sets $A, B \in \mathcal{F}$, we have $|A \cap B| \geq t$. If $t = 1$, then we write that $\mathcal{F}$ is *intersecting.*

1.2 History

The maximum size of an antichain in $2^{[n]}$ was determined by Sperner [9].

Theorem 1.3 (Sperner). *Let $\mathcal{F} \subseteq 2^{[n]}$ be an antichain. Then,*

$$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Furthermore, equality holds only if $\mathcal{F}$ is one of the largest layers in the Boolean lattice $2^{[n]}$.

Sperner's theorem was extended to k-Sperner families by Erdős [2].

Theorem 1.4 (Erdős). *The maximum-size k-Sperner family $\mathcal{F} \subseteq 2^{[n]}$ is the union of the largest k layers in the Boolean lattice $2^{[n]}$.*

A different extension of Sperner's theorem was given by Milner [8]. Milner additionally required the family $\mathcal{F}$ to be t-intersecting.

Theorem 1.5 (Milner). *If $\mathcal{F} \subseteq 2^{[n]}$ is a t-intersecting antichain, then*

$$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n+t+1}{2} \rfloor}.$$

In a different direction, Frankl [3] determined the maximum size of an intersecting k-Sperner family. Different proofs were given by Gerbner [5] and by Gerbner, Methuku and Tompkins [6].

Theorem 1.6 (Frankl). *Let $\mathcal{F} \subseteq 2^{[n]}$ be an intersecting, k-Sperner family. Then,*

$$|\mathcal{F}| \leq \begin{cases} \sum_{i=\frac{n+1}{2}}^{\frac{n+1}{2}+k-1} \binom{n}{i}, & \text{if } n \text{ is odd,} \\ \binom{n-1}{\frac{n}{2}-1} + \sum_{i=\frac{n}{2}+1}^{\frac{n}{2}+k-1} \binom{n}{i} + \binom{n-1}{\frac{n}{2}+k}, & \text{if } n \text{ is even.} \end{cases}$$

Furthermore, if n is odd, equality holds only if

$$\mathcal{F} = \binom{[n]}{\lfloor \frac{n}{2} \rfloor + 1} \cup \binom{[n]}{\lfloor \frac{n}{2} \rfloor + 2} \cup \ldots \cup \binom{[n]}{\lfloor \frac{n}{2} \rfloor + k},$$

while if n is even and $k > 1$, equality holds only if for some $x \in [n]$,

$$\mathcal{F} = \left\{ F \in \binom{[n]}{\frac{n}{2}} : x \in F \right\} \cup \binom{[n]}{\frac{n}{2} + 1} \cup \ldots \cup \binom{[n]}{\frac{n}{2} + k - 1} \cup \left\{ F \in \binom{[n]}{\frac{n}{2} + k} : x \notin F \right\}.$$

A common generalization of the theorems of Milner and Frankl would be to determine the maximum size of a t-intersecting, k-Sperner family.

Frankl [4] proposed conjectures on the maximum size of a t-intersecting k-Sperner family $\mathcal{F} \subset 2^{[n]}$ and made some progress towards proving these conjectures. The conjectured extremal family depends on the parity of $n + t$.

In the case when $n + t$ is even, the conjectured maximum size of a t-intersecting, k-Sperner family is very easy to describe.

Conjecture 1.7 (Frankl)**.** *If $n+t$ is even, $n > t$, and $\mathcal{F} \subseteq 2^{[n]}$ is a t-intersecting, k-Sperner family, then*

$$|\mathcal{F}| \leq \sum_{i=0}^{k-1} \binom{n}{\frac{n+t}{2} + i}.$$

Conjecture 1.7 is clearly tight if true, as evidenced by the family $\bigcup_{i=0}^{k-1} \binom{[n]}{\frac{n+t}{2}+i}$.

The conjectured extremal families do not have such a simple structure when $n + t$ is odd. We construct two plausible candidates for the maximum size t-intersecting, k-Sperner family:

$$\mathcal{A}(t,k) = \left\{ F \in \binom{[n]}{\frac{n+t-1}{2}} : n \notin F \right\} \cup \left\{ A : \frac{n+t-1}{2} + 1 \leq |A| \leq \frac{n+t-1}{2} + (k-1) \right\}.$$

$$\mathcal{B}(t,k) = \left\{ F \in \binom{[n]}{\frac{n+t-1}{2}} : [1,t] \in F \right\} \cup \left\{ A : \frac{n+t-1}{2} + 1 \leq |A| \leq \frac{n+t-1}{2} + (k-1) \right\}$$
$$\cup \left(\left\{ B : |B| = \frac{n+t-1}{2} + k \right\} \setminus \left\{ B : |B| = \frac{n+t-1}{2} + k, [1,t] \in B \right\} \right).$$

It is not hard to show that $|\mathcal{B}(t,k)| \gg |\mathcal{A}(t,k)|$ for n sufficiently large (in terms of k and t). However, it may be checked by computer that $\mathcal{A}(t,k)$ is optimal for small values of n and specific choices of t and k, for example $t = 2$ and $k = 2$. We conjecture that $\mathcal{B}(t,k)$ is the largest such family when n is sufficiently large.

Conjecture 1.8. *There exists a positive integer $n_0 = n_0(k,t)$ such that if $n+t$ is odd, $n > n_0$, and $\mathcal{F} \subseteq 2^{[n]}$ is a t-intersecting, k-Sperner family, then*

$$|\mathcal{F}| \le |\mathcal{B}(t,k)| = \binom{n-t}{\frac{n-t-1}{2}} + \sum_{i=1}^{k} \binom{n}{\frac{n+t-1}{2}+i} - \binom{n-t}{\frac{n-t-1}{2}+k}.$$

Frankl [4] more modestly conjectures the following (Frankl's conjecture is formulated for s-union families rather than t-intersecting families, but our formulation is equivalent to Frankl's after taking complements).

Conjecture 1.9 (Frankl). *Let $g(n,t,k) := \max\{|\mathcal{G}| - |\Delta_{\frac{n-t+1}{2}-k}(\mathcal{G})| : \ \mathcal{G} \subset \binom{[n]}{\frac{n-t+1}{2}}$ is intersecting$\}$. Then, if $n+t$ is odd and $\mathcal{F}$ is a t-intersecting, k-Sperner family, then*

$$|\mathcal{F}| \le g(n,t,k) + \sum_{i=1}^{k} \binom{n}{\frac{n+t-1}{2}+i}.$$

Note that Conjecture 1.8 can be interpreted as a strengthening of Conjecture 1.9, in that additionally there is a conjecture for the value of the function $g(n,t,k)$ for sufficiently large n. The connection may be made more apparent by noting that, after taking complements, we may equivalently define $g(n,t,k) := \max\{|\mathcal{G}| - |\nabla_{\frac{n+t-1}{2}+k}(\mathcal{G})| : \ \mathcal{G} \subset \binom{[n]}{\frac{n+t-1}{2}}$ is t-intersecting$\}$.

1.3 New Results

Let us mention that Frankl proved Conjecture 1.7 when $t \ge n - O(\sqrt{n})$. We settle Conjecture 1.7 if t is fixed and n is sufficiently large.

Theorem 1.10. *Let t and k be positive integers, and suppose that $n+t$ is even with $t \le n$, and n is large enough. If $\mathcal{F} \subseteq 2^{[n]}$ is a t-intersecting k-Sperner family, then*

$$|\mathcal{F}| \le \binom{[n]}{\frac{n+t}{2}} + \ldots + \binom{[n]}{\frac{n+t}{2}+k-1}.$$

2 Sketch of the Proof of Theorem 1.10

The proof of Theorem 1.10 consists of three parts. The first part is a so-called "push-to-the-middle" argument. By proving this part, one obtains that there exists a maximum size t-intersecting k-Sperner family that contains sets only of cardinality between $\frac{n+t}{2}-(k-1)$ and $\frac{n+t}{2}+2(k-1)$. This is achieved in two steps, none of which uses the assumption $n+t$ even, so this part of the proof can be applied in arguments for the $n+t$ odd case. The two steps are formulated in the following two lemmas, the first of which applies Katona's shadow t-intersection theorem [7].

Lemma 2.1. *Let $\mathcal{F} \subseteq 2^{[n]}$ be a t-intersecting and k-Sperner family, where $n+t$ is even. Then there exists a t-intersecting k-Sperner family $\mathcal{G} \subseteq 2^{[n]}$ with $|\mathcal{G}| \geq |\mathcal{F}|$ and $\min\{|G| : G \in \mathcal{G}\} \geq \frac{n+t}{2} - (k-1)$.*

Lemma 2.2. *If $\mathcal{F} \subseteq 2^{[n]}$ is a t-intersecting k-Sperner family with $\min\{|F| : F \in \mathcal{F}\} = \frac{n+t}{2} - c$, then there exists a t-intersecting k-Sperner family $\mathcal{F}' \subseteq 2^{[n]}$ with $|\mathcal{F}| \leq |\mathcal{F}'|$, and*

$$\min\{|F| : F \in \mathcal{F}\} = \min\{|F'| : F' \in \mathcal{F}'\} \quad \text{and} \quad \max\{|F'| : F' \in \mathcal{F}'\} \leq \frac{n+t}{2} + c + k - 1.$$

The other two parts of the proof applies Katona's cycle method. We need some definitions. Let σ be a cyclic permutation of $[n]$ and $\mathcal{F}_\sigma$ be the subfamily of those sets in $\mathcal{F}$ that form an interval in σ. Note that there are $(n-1)!$ choices for σ. For a set G, let $w(G) = \binom{n}{|G|}$ and $w(\mathcal{G}) = \sum_{G \in \mathcal{G}} w(G)$. We define m as $m := \frac{n+t}{2} - \min\{|F| : F \in \mathcal{F}''\}$. By the above discussions, we have $0 \leq m \leq k-1$. If $m = 0$ then $\mathcal{F}$ has the required structure, hence we assume $m > 0$. The second and most delicate part of the proof is the following lemma that determines the maximum weight of a t-intersecting k-Sperner family on the cycle.

Lemma 2.3. *Suppose $n+t$ is even with $t \leq n$ and n is large enough. For every cyclic permutation σ and t-intersecting k-Sperner family $\mathcal{F} \subseteq \bigcup_{i=\frac{n+t}{2}-m}^{\frac{n+t}{2}+k-1+m} \binom{[n]}{i}$, we have $w(\mathcal{F}_\sigma) \leq n \sum_{i=0}^{k-1} \binom{n}{\frac{n+t}{2}+i}$.*

Before giving more insight on the proof of Lemma 2.3, let us show how Lemma 2.3 implies Theorem 1.10 that is the part of the proof.

Proof of Theorem 1.10 using Lemma 2.3. As mentioned in the last paragraph of the previous subsection, by Theorem 2.1 and Lemma 2.2, we can assume that $\mathcal{F} \subseteq \bigcup_{i=\frac{n+t}{2}-m}^{\frac{n+t}{2}+k-1+m} \binom{[n]}{i}$ holds. Then using Lemma 2.3 we have:

$$\sum_{\sigma} \sum_{F \in \mathcal{F}_\sigma} w(F) \leq (n-1)! \cdot n \sum_{i=0}^{k-1} \binom{n}{\frac{n+t}{2}+i} = n! \cdot \sum_{i=0}^{k-1} \binom{n}{\frac{n+t}{2}+i}.$$

From the other side,

$$\sum_{\sigma} \sum_{F \in \mathcal{F}_\sigma} w(F) = \sum_{F \in \mathcal{F}} |F|!(n-|F|)! \binom{n}{|F|} = n!|\mathcal{F}|,$$

which implies the required upper bound on $|\mathcal{F}|$. $\square$

Let us return to the proof of Lemma 2.3. The extremal family of the lemma consists of all kn intervals with size between $\frac{n+t}{2}$ and $\frac{n+t}{2} + k - 1$. Also, any k-Sperner family on the cycle may contain at most kn intervals, but those that have cardinality smaller than $\frac{n+t}{2}$ have larger weight than the intervals of the extremal family. To compensate for these, we need to show that if such intervals exist in a t-intersecting K-Sperner family $\mathcal{G}$, then $\mathcal{G}$ must contain intervals that are too large, larger than $\frac{n+t}{2}+k-1$ and therefore have smaller weight than all intervals of the extremal family. (And we have to make sure that the loss of weight is more than the gain.) This is done based on the following simple observation. For a cyclic permutation σ and an interval G define $\overline{G}^t$ as the complement of G together with the (counterclockwise) leftmost $\lfloor \frac{t}{2} \rfloor$ and rightmost $\lceil \frac{t}{2} \rceil$ elements of G with respect to σ. For a family $\mathcal{G}$ of intervals, let $\overline{\mathcal{G}}^t = \{\overline{G}^t : G \in \mathcal{G}\}$. Observe that $|G \cap \overline{G}^t| = t$ and the two endpoints of $\overline{G}^t$ belong to $G \cap \overline{G}^t$. Therefore, if $\mathcal{G}$ is t-intersecting, then for any $G \in \mathcal{G}$ no proper subinterval H of $\overline{G}^t$ belongs to $\mathcal{G}$. $|G| = \frac{n+t}{2} - c$ implies $|\overline{G}^t| = n - (\frac{n+t}{2} - c) + t = \frac{n+t}{2} + c$, so if such a G belongs to $\mathcal{G}$, then many intervals of size close $\frac{n+t}{2}$ cannot belong to $\mathcal{G}$.

The details how this is used to derive Lemma 2.3 can be found in [1].

References

[1] József Balogh, William B Linz, and Balázs Patkós. On the sizes of t-intersecting k-chain-free families. *arXiv preprint arXiv:2209.01656*, 2022.

[2] Paul Erdős. On a lemma of Littlewood and Offord. *Bull. Amer. Math. Soc.*, 51:898–902, 1945.

[3] Peter Frankl. Canonical antichains on the circle and applications. *SIAM Journal on Discrete Mathematics*, 3(3):355–363, 1990.

[4] Peter Frankl. Analogues of Milner's theorem for families without long chains and of vector spaces. *European Journal of Combinatorics*, 93:103279, 2021.

[5] Dániel Gerbner. Profile polytopes of some classes of families. *Combinatorica*, 33:199–216, 2013.

[6] Dániel Gerbner, Abhishek Methuku, and Casey Tompkins. Intersecting P-free families. *Journal of Combinatorial Theory, Series A*, 151:61–83, 2017.

[7] Gyula Katona. Intersection theorems for systems of finite sets. *Acta Mathematica Academiae Scientiarum Hungaricae*, 15(3-4):329–337, 1964.

[8] Eric C Milner. A combinatorial theorem on systems of sets. *Journal of the London Mathematical Society*, 1(1):204–206, 1968.

[9] Emanuel Sperner. Ein satz über untermengen einer endlichen menge. *Mathematische Zeitschrift*, 27(1):544–548, 1928.