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Directed graphs without rainbow triangles[*]

(Extended abstract)

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Abstract

One of the most fundamental questions in graph theory is Mantel's theorem which determines the maximum number of edges in a triangle-free graph of a given order. Recently a colorful variant of this problem has been solved. In such a variant we consider c graphs on a common vertex set, thinking of each graph as edges in a distinct color, and want to determine the smallest number of edges in each color which guarantees the existence of a rainbow triangle. Here, we solve the analogous problem for directed graphs without rainbow triangles, either directed or transitive, for any number of colors. The constructions and proofs essentially differ for $c = 3$ and $c \geq 4$ and the type of the forbidden triangle.

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1 Introduction

A cornerstone of the extremal graph theory is Mantel's theorem from 1907 which determines the maximum possible number of edges in a triangle-free graph of a given order. Its natural generalization, known as a Turán problem, is to determine the maximum possible number

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of edges in an n-vertex graph not containing a given graph F as a subgraph. This is an often studied concept in graph theory with many important results, open problems, and generalizations to various other discrete settings.

In a rainbow version of the Turán problem, for a graph F and an integer c we consider c graphs $G_1, G_2, \ldots, G_c$ on the same set of vertices and ask for the maximum possible number of edges in each graph avoiding the appearance of a copy of F having at most one edge from each graph. In other words, for every i we color edges of G_i in color i and forbid all copies of F having non-repeated colors, so-called rainbow copies. Note that if all G_i are exactly the same, then the existence of a rainbow copy of F is equivalent to the existence of a non-colored copy of F, therefore any bound for the rainbow version gives also a bound for the Turán problem.

When the forbidden graph F is a triangle, Keevash, Saks, Sudakov and Verstraëte [10] showed that for $c \geq 4$ colors the best possible number of edges in each color without having a rainbow triangle is equal to $\frac{1}{4}n^2$. This is achieved in the balanced complete bipartite graph (the same in each color) as in Mantel's theorem. Surprisingly, Magnant [11] provided a construction showing that for 3 colors the answer is different. Recently, Aharoni, DeVos, González, Montejano and Šámal [1], answering a question of Diwan and Mubayi [4], proved that for 3 colors the optimal bound is $\left(\frac{26-2\sqrt{7}}{81}\right)n^2 \approx 0.2557n^2$.

Later, Frankl [6] made a conjecture on the optimal bound for the product of the number of edges in each of the 3 colors without having a rainbow triangle. This was disproved by Frankl, Győri, Hel, Lv, Salia, Tomkins, Varga and Zhu [7]. Finally, Falgas-Ravry, Markström and Räty [5] completely determined the triples of the asymptotic number of edges in each color that force an existence of a rainbow triangle. Similar problems were also considered for other rainbow structures than triangles, for instance for paths [2], color-critical graphs [3], or spanning subgraphs [8, 9].

Here, we consider the problem in the setting of directed graphs and solve it for any number of colors when a transitive triangle or a directed triangle is the forbidden rainbow graph. It occurs that for both kinds of triangles and at least 4 colors, the maximum number of edges in each graph is attained when each of them is the same graph maximizing the number of edges without creating the forbidden triangle. While for 3 colors, the behavior is completely different. In case of a transitive triangle the construction is as in the result of Aharoni et al. [1] with all edges replaced by arcs in both ways. While for a directed triangle the construction is again significantly different.

In the next section we introduce the used notation, while in Section 3 we state our theorems and sketch their proofs. If a rainbow directed triangle is forbidden and there are at least 4 colors we show that the optimal asymptotic value of the maximal number of edges in each color follows from the bound on the total number of colored edges (Theorem 1), while for 3 colors we show that the optimal bound follows from the bound on the sum of the number of edges in any two colors (Theorem 3). Using similar case distinction and generalizations, we solve the rainbow Turán problem for a transitive triangle and at least 4 colors by Theorem 5 and finally using Theorem 7 we prove the optimal bound in the 3 colors case.

2 Notation

A *directed graph* G is a pair $(V(G), E(G))$, where $V(G)$ is a set of vertices and $E(G)$ is a set of pairs of distinct vertices. In particular, G does not contain loops or multiple edges. For shortening, we denote $e(G)$ as $|E(G)|$. For two vertices $u, v \in V(G)$, we write uv to denote the edge $(u, v) \in E(G)$. We refer jointly to edges uv and vu as edges *between* u and v. If $uv, vu \in E(G)$ we say that u and v are connected with a *double edge*. A *directed triangle* is a directed graph on the vertex set $\{u, v, w\}$ with edges uv, vw and wu, while a *transitive triangle* is a directed graph on the vertex set $\{u, v, w\}$ with edges uv, vw and uw. We denote by $[c]$ the set of positive integers $\{1, 2, \dots, c\}$.

Having an ordered set of directed graphs $G := (G_1, G_2, \dots, G_c)$ on a common vertex set $V(G)$, we consider the edge set of each graph G_i, for $i \in [c]$, as *edges* of G in color i. For a directed graph F, we say that G contains a rainbow copy of F if $V(F) \subset V(G)$ and there is a coloring φ of the edges of F into distinct colors such that $e \in E(G_{\varphi(e)})$ for every edge $e \in E(F)$.

3 Our results

We start with forbidding a directed triangle in the setting with at least 4 colors and prove the following theorem with its immediate corollary.

Theorem 1. *Let $c \geq 4$ and $G_1, G_2, \dots, G_c$ be directed graphs on a common set of n vertices. If $\sum_{i=1}^{c} e(G_i) > c\lfloor \frac{n^2}{2} \rfloor$, then there exists a rainbow directed triangle.*

Corollary 2. *Let $c \geq 4$ and $G_1, G_2, \dots, G_c$ be directed graphs on a common set of n vertices. If $\min_{i \in [c]} e(G_i) > \lfloor \frac{n^2}{2} \rfloor$, then there exists a rainbow directed triangle.*

The bounds provided above are the best possible. To observe this, consider each graph G_i for $i \in [c]$ as the same directed graph having $\lfloor \frac{n^2}{2} \rfloor$ edges constructed by replacing each edge of a complete bipartite graphs $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ by a double edge.

Sketch of the proof of Theorem 1. Suppose (for a contradiction) that Theorem 1 is false and choose a counterexample with the smallest number of vertices n. Firstly, we prove that there do not exist two vertices u and v connected with a double edge in two colors. This is obtained by considering how many edges can be between any vertex and vertices u and v, and using the assumed minimality. It implies that between any two vertices, there are at most $c + 1$ edges, where only one of them can be a double edge. Then, using a similar approach and careful counting, we show that the vertices connected with $c + 1$ edges can create only directed paths. Consequently, there are fewer edges than in our extremal construction, which leads to a contradiction. $\square$

In case of 3 colors, Theorem 1 cannot hold, because if G_1 and G_2 are graphs with double edges between each pair of vertices and G_3 is an empty graph, then $\sum_{i=1}^{3} e(G_i) = 2n(n-1) > 3\lfloor \frac{n^2}{2} \rfloor$ and there is no rainbow directed triangle. In this case we prove the following theorem and its corollary.

Theorem 3. *Let G_1, G_2, G_3 be three directed graphs on a common set of n vertices. If $e(G_i) + e(G_j) > \frac{10}{9}n^2 + 2n$ for $1 \le i < j \le 3$, then there exists a rainbow directed triangle.*

Corollary 4. *Let G_1, G_2, G_3 be three directed graphs on a common set of n vertices. If $\min_{i \in [c]} e(G_i) > \frac{5}{9}n^2 + n$, then there exists a rainbow directed triangle.*

To see that the bound in Theorem 3 is asymptotically the best possible, consider three sets of $n/3$ vertices A_1, A_2 and A_3, and for each $i \in [3]$ let $E(G_i)$ contain all double edges inside A_j for $j \neq i$, as well as edges from A_1 to A_2, from A_1 to A_3, and from A_2 to A_3. This is depicted in Figure 1. This construction has approximately $\frac{5}{9}n^2$ edges in each color and does not contain a rainbow directed triangle.

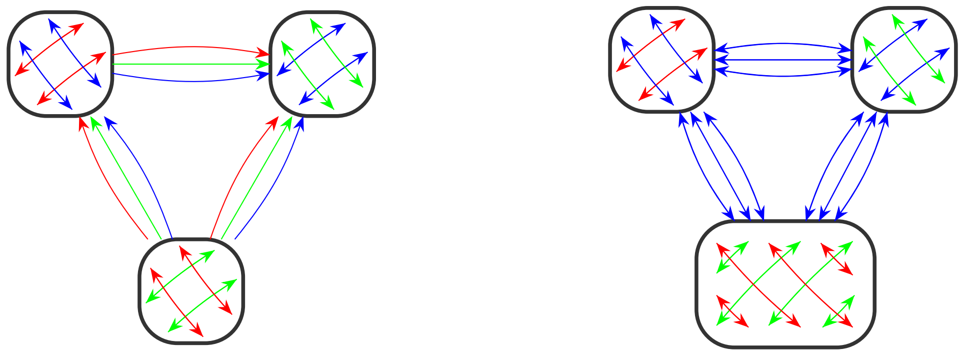

Figure 1: The extremal construction for 3 colors and forbidden rainbow directed triangle (on the left) and rainbow transitive triangle (on the right).

Sketch of the proof of Theorem 3. Consider a counterexample $G = (G_1, G_2, G_3)$ with the smallest number of vertices. Firstly, using the assumed minimality, we prove that there is no pair of vertices connected with double edges in all three colors. Using this, we split the vertices of G into disjoint sets. Consider a set X of vertices forming a maximal matching of double edges in two colors (there might be a single edge in the third color). Then, on the vertices $V(G)\setminus X$ consider a set Y forming a maximal matching consisting of pairs of vertices connected with 4 edges. Next, on the vertices $V(G) \setminus (X \cup Y)$ consider a set Z forming a maximal matching consisting of pairs of vertices connected by a double edge in one color and an edge in a different color. Finally, let T be a set of vertices forming a maximal matching on the vertices $V(G) \setminus (X \cup Y \cup Z)$ consisting of pairs of vertices connected with 3 edges. From the maximality of Z pairs in T are connected by a single edge in each color. From the maximality of T, all pairs of vertices in $D = V(G)\setminus(X \cup Y \cup Z \cup T)$ are connected by at most 2 edges.

Having such partitioning, we bound the total number of edges and the number of edges in each pair of colors in terms of the sizes of the defined sets. It is possible, as the number of edges between the vertices in respective sets are either limited by the definition of the sets or by the possibility of creating a rainbow directed triangle. Moreover, to include the fact that having many edges between set D and sets X, Y and Z is limiting the number of edges inside those sets, we use additional Turán-type bounds on an auxiliary graph.

Altogether, we obtain an optimization problem on 4 variables with linear or quadratic functions, which has a unique solution giving exactly the structure depicted in Figure 1. □

We continue with forbidding a transitive triangle in the setting with at least four colors and prove the following theorem.

Theorem 5. *Let $c \geq 4$ and $G_1, G_2, \ldots, G_c$ be directed graphs on a common set of n vertices. If $\sum_{i=1}^{c} e(G_i) > c\lfloor \frac{n^2}{2} \rfloor$, then there exists a rainbow transitive triangle.*

Similarly to the case of a forbidden directed triangle, this theorem easily implies the following corollary.

Corollary 6. *Let $c \geq 4$ and $G_1, G_2, \ldots, G_c$ be directed graphs on a common set of n vertices. If $\min_{i\in[c]} e(G_i) > \lfloor \frac{n^2}{2} \rfloor$, then there exists a rainbow transitive triangle.*

The bounds provided in Theorem 5 and Corollary 6 are the best possible. To observe this, consider, like in the case of a forbidden rainbow directed triangle, each graph G_i for $i \in [c]$ as the same directed graph constructed by replacing each edge of a complete bipartite graphs $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ by a double edge. In this construction, there is no directed triangle and $e(G_i) = \lfloor \frac{n^2}{2} \rfloor$ for each $i \in [c]$.

Sketch of the proof of Theorem 5. Consider a hypothetical counterexample with the smallest number of vertices. Firstly, similarly as in the proof of Theorem 1, we prove that it does not contain a double edge in two colors. Having this limitation, we prove that it cannot contain any double edge at all. This immediately leads to a contradiction with the assumed number of edges. □

Similarly as in the case of the forbidden rainbow directed triangle, in case of 3 colors Theorem 5 cannot hold. In this case we prove the following theorem and its corollary.

Theorem 7. *Let G_1, G_2, G_3 be three directed graphs on a common set of n vertices. If for every $1 \leq i < j \leq 3$ it holds $e(G_i) + e(G_j) > \left(\frac{104-8\sqrt{7}}{81}\right) n^2 + 2n$, then there exists a rainbow transitive triangle.*

Corollary 8. *Let G_1, G_2, G_3 be three directed graphs on a common set of n vertices. If $\min_{i\in[c]} e(G_i) > \left(\frac{52-4\sqrt{7}}{81}\right) n^2 + n$, then there exists a rainbow transitive triangle.*

The bound in Theorem 7 and Corollary 8 is asymptotically optimal in the sense that it is not possible to prove analogous statements with lower constants by the n^2 term. This is a consequence of the construction obtained from the optimal construction of Aharoni et al. [1] for the forbidden rainbow triangle by replacing all edges by double edges, as depicted in Figure 1.

Sketch of the proof of Theorem 7. We follow the idea of the proof from [1] for the forbidden rainbow triangle. It is not straightforward that those bounds translate to the directed setting as here it is possible to contain rainbow triangles as long as they are directed. Nevertheless, we prove bounds that give the same optimization problem (multiplied by a factor of 2). Thus, [1] implies the desired bound. □

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