Graphs without a rainbow path of length 3[*]

(Extended abstract)

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Abstract

In 1959 Erdős and Gallai proved the asymptotically optimal bound for the maximum number of edges in graphs not containing a path of a fixed length. We investigate a rainbow version of the theorem, in which one considers $k \geq 1$ graphs on a common set of vertices not creating a path having edges from different graphs and asks for the maximum number of edges in each graph. We prove the asymptotically optimal bound in the case of a path on three edges and any $k \geq 1$.

1 Introduction

A classical problem in graph theory is to determine the Turán number of a graph F, i.e., the maximum possible number of edges in graphs not containing a particular forbidden structure F as a subgraph. The notable results are exact solutions for a triangle by Mantel [16] and for a complete graph by Turán [17], and an asymptotically optimal bound for any non-bipartite graph by Erdős and Stone [6]. Not much is known for bipartite graphs, but the case of a path was solved asymptotically by Erdős and Gallai [5] in 1951, while in 1975 Faudree and Schelp [7] provided an exact solution.

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There are many possible ways to define a rainbow version of the problem. In our work we concentrate on a rainbow version without any additional assumptions and when the number of edges in each color is maximized. Formally, for a graph F and a positive integer k we consider k graphs $G_1, G_2, \ldots, G_k$ on the same set of vertices and ask for the maximum possible number of edges in each graph avoiding appearance of a copy of F having at most one edge from each graph. In other words, for every i we color edges of G_i in color i (in particular it means that two vertices can be connected by edges in many colors) and forbid all copies of F having non-repeated colors, so-called rainbow copies.

When the forbidden rainbow graph F is a triangle, it follows from a result of Keevash, Saks, Sudakov and Verstraëte [13] that for $k \geq 4$ colors the best possible number of edges in each color without having a rainbow triangle is equal to $\frac{1}{4}n^2$. This is achieved in the balanced complete bipartite graph (the same in each color) as in Mantel's theorem. Surprisingly, Magnant [15] provided a construction showing that for 3 colors the answer is different. Later, Aharoni, DeVos, de la Maza, Montejano and Šámal [1], answering a question of Diwan and Mubayi [4], proved that in this case the asymptotically optimal bound is $\left(\frac{26-2\sqrt{7}}{81} \right) n^2 \approx 0.2557n^2$. They also asked for similar theorems for bigger cliques, other graphs and different color patterns (in this setting some results were proven in [3] and [14]). Recently, Falgas-Ravry, Markström and Räty [8] completely determined the triples of the asymptotic number of edges in each color that force an existence of a rainbow triangle. Similar problems, but where one maximizes other functions of the number of edges (instead of the number of edges in each color), were considered e.g. in [2, 9, 11, 12].

2 The main result

In our work we consider an arbitrary fixed number of colors $k \geq 1$ and we aim to maximize the number of edges in each color avoiding a rainbow path of length 3. The bound obtained is asymptotically tight.

Theorem 1. *For every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $k \geq 1$ and graphs $G_1, G_2, \ldots, G_k$ on a common set of n vertices, each graph having at least $(f(k)+\varepsilon)\frac{n^2}{2}$ edges, where*

$$f(k) = \begin{cases} \lceil \frac{k}{2} \rceil^{-2} & \text{for } k \leq 6, \\ \frac{1}{2k-1} & \text{for } k \geq 7, \end{cases}$$

there exists a rainbow path with 3 edges. Moreover, the above bound on the number of edges is asymptotically optimal for each $k \geq 1$.

We note that in [12] the same forbidden structure is considered, i.e., a rainbow path of length 3, however when one aims to maximize the product of the number of edges in each color and there are only 3 or 4 colors. While the result for 4 colors provided there implies our result for 4 colors, the results for 3 colors are independent of each other.

In order to avoid struggling with the lower-order error terms and to obtain a structure easier to handle, we rewrite Theorem 1 to a bit different setting. Assuming that Theorem 1

does not hold we obtain an arbitrarily large counterexample with at least $(f(k) + \varepsilon)\frac{n^2}{2}$ edges in each color and without a rainbow path with 3 edges. Using colored graph removal lemma [10] we remove all rainbow walks with 3 edges by deleting at most $\frac{1}{4}\varepsilon n^2$ edges in each color. Then, we add all possible edges without creating rainbow walks with 3 edges. Finally, we group all the vertices into clusters based on the colors on the incident edges. Note that if there is an edge between two clusters (or inside one), then all the vertices between these clusters (or inside this cluster) can be connected by edges in the same color without creating a rainbow walk of length 3. Thus between clusters (and inside them) in each color we have all possible edges or none. Additionally, notice that vertices in a cluster incident to only one or two colors can be all connected by edges in those colors, while vertices incident to more than 2 colors need to form an independent set. Therefore, Theorem 1 can be stated in an equivalent form for such kind of clustered graphs.

Definition 2. For any integer $k \geq 1$ a *clustered graph for k colors* is an edge-colored weighted graph on $\binom{k}{2} + k + 1$ vertices with vertex weights $b_{ij} = b_{ji}$ for $1 \leq i < j \leq k$, a_i for $i \in [k]$ and x, in which

- $x \geq 0$, $a_i \geq 0$ for $i \in [k]$ and $b_{ij} \geq 0$ for every $1 \leq i < j \leq k$,

- $$\sum_{1 \leq i < j \leq k} b_{ij} + \sum_{1 \leq i \leq k} a_i + x = 1,$$

- for every $i \in [k]$ the vertex of weight a_i is connected in color i with itself, the vertex of weight x and all the vertices of weights b_{ip} for $p \neq i$,

- for every $1 \leq i < j \leq k$ each vertex of weight b_{ij} is connected in colors i and j with itself,

- there are no other edges.

The vertex of weight b_{ij} represents the cluster of $b_{ij}n$ vertices incident to edges colored i and j, the vertex of weight a_i – the cluster of $a_i n$ vertices incident only to edges colored i, and x represents the remaining vertices. Clusters for b_{ij} and a_i are cliques in appropriate colors, while cluster for x is an independent set. This is depicted for $k = 3$ in Figure 1.
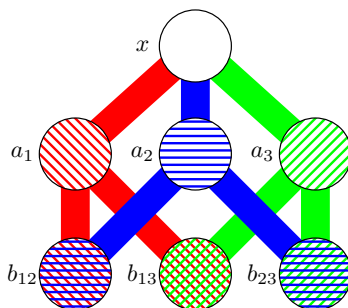

Figure 1: Representation of clusters for $k = 3$.

From the definition of a clustered graph it follows that the *density of edges* in color $i \in [k]$ in a clustered graph G is the number $d_i(G) \in [0, 1]$ equal to

$$d_i(G) = a_i^2 + \left(\sum_{j \in [k] \setminus \{i\}} b_{ij}^2 \right) + \left(2 \sum_{j \in [k] \setminus \{i\}} a_i b_{ij} \right) + 2a_i x.$$

The equivalent version of Theorem 1 for clustered graphs is the following.

Theorem 3. *For every integer $k \geq 1$, if G is a clustered graph for k colors, then*

$$\min_{i \in [k]} d_i(G) \leq f(k), \ \ where \ f(k) = \begin{cases} \lceil \frac{k}{2} \rceil^{-2} & for \ k \leq 6, \\ \frac{1}{2k-1} & for \ k \geq 7. \end{cases}$$

Theorem 1 follows from Theorem 3, because a possible counterexample leads to a graph with density of edges in each color at least $(f(k) + \frac{1}{2}\varepsilon)\frac{n^2}{2}$ and clusters of vertices behaving as weighted vertices of a related clustered graph. Dividing each cluster size by n we obtain a clustered graph with density of edges in each color at least $f(k) + \frac{1}{2}\varepsilon$, which contradicts Theorem 3. Note that also Theorem 1 implies Theorem 3 as any clustered graph G contradicting Theorem 3 having $d_i(G) \geq f(k) + 2\varepsilon$ for each $i \in [k]$ and some $\varepsilon > 0$ leads for any appropriately large n to a graph on n vertices with at last $(f(k) + \varepsilon)\frac{n^2}{2}$ edges in each color and no rainbow path with 3 edges, which contradicts Theorem 1.

The bound provided in Theorem 3 is tight for every integer $k \geq 1$, because it is possible to construct a clustered graph for k colors G such that $\min_{i \in [k]} d_i(G) = f(k)$:

— for $k = 1$ let $a_1 = 1$;
— for $k = 2$ let $b_{12} = 1$;
— for $k = 3$ let $b_{12} = b_{13} = \frac{1}{2}$;
— for $k = 4$ let $b_{12} = b_{34} = \frac{1}{2}$;
— for $k = 5$ let $b_{12} = b_{34} = b_{15} = \frac{1}{3}$;
— for $k = 6$ let $b_{12} = b_{34} = b_{56} = \frac{1}{3}$;
— for $k = 5$ or $k \geq 7$ let $a_i = \frac{1}{2k-1}$ for each $i \in [k]$, $x = \frac{k-1}{2k-1}$.

In each case the remaining weights are equal to 0.

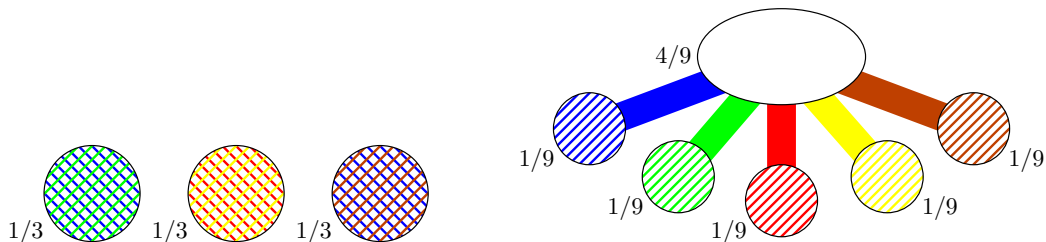For $k = 5$ there are two different types of constructions. They are depicted in Figure 2.

Figure 2: Two possible types of extremal constructions for $k = 5$.

3   Outline of the proof

Theorem 3 is proven by induction. The theorem is trivial for $k \in \{1, 2\}$ as then $f(k) = 1$. Let us fix the smallest $k \geq 3$ for which the theorem does not hold. Take a clustered graph for k colors G maximizing the value of $\min_{i \in [k]} d_i(G)$ and, among such, maximizing the density of edges in any color. The assumption that Theorem 3 does not hold implies that $d_i(G) > f(k)$ for every $i \in [k]$. This and the maximality of G enable to show claims on the weights of the vertices of G, which will lead to a contradiction for each value of $k \geq 3$.

To find many useful bounds on the weights, we introduce an operation of removing and adding weights in a clustered graph for k colors. Intuitively, we remove tiny weights from some of the vertices of positive weight and add them to different vertices. From the maximality of G, such operation cannot enlarge the density of edges in each color, so the density of edges in at least one color needs to drop down (or the densities of edges in every color remain the same). Due to different extremal constructions, we need to consider three main cases: $k \in \{3, 4\}$, $k \in \{5, 6\}$ and $k \geq 7$. Let us denote $d_i = d_i(G)$, $b_i = \sum_{j \in [k], j \neq i} b_{ij}$, and $c_i = a_i + b_i + x$.

In the case of $k = 3$ our conjectured clustered graph G satisfies $\min_{i \in [3]} d_i > \frac{1}{4}$, which implies a simple lower bound on c_i. We prove that for some $i \in [3]$ we have $a_i = 0$ (without loss of generality $a_3 = 0$). By contradiction, if every $a_i > 0$, we can remove appropriate weight from each vertex of weight a_i and add the removed weights to all vertices. It implies, using the maximality of G, a better lower bound for some c_i, say c_3. Now we consider two cases: $x \neq 0$ and $x = 0$. In the former one we find a lower bound on $\sum_{i \in [3]} a_i$ by removing suitable weight from the vertex of weight x and adding weights to each vertex of weight a_i. Together with bounds on $c_i, i \in [3]$ it gives a contradiction. While in the latter case, we show first that $b_{12} > 0$ and removing appropriate weight from the vertex of weight b_{12} and adding weights to each vertex of the graph leads to a contradiction. Once we know that $a_3 = 0$, using the technique of removing and adding weights, we show that $x = 0$ and that $d_3 \leq \min\{d_1, d_2\}$. Then by removing suitable weight from b_{12} and adding weights to a_1 and a_2, we obtain a contradiction which finishes the proof of Theorem 3 for $k = 3$. The case $k = 4$ is a simple corollary of the theorem for $k = 3$ since $f(4) = f(3)$.

The proof of Theorem 3 for $k = 5$ relies on similar techniques. However, as there are two types of constructions, it requires more careful estimations and thus additional bounds and considering more cases. In particular, we prove a different lower bound for b_{ij} when it is positive and a bound on b_i when $a_i = 0$. Having this, depending on the number of $i \in [5]$ such that $a_i = 0$, we bound the sum of all c_i and obtain a contradiction in each case. The proof for $k = 6$ is a simple consequence of the result for $k = 5$.

In the case of $k \geq 7$ we first show that x must be positive. Then we separately prove cases $k = 7$, $k = 8$ and $k \geq 9$. In the first one we sum up lower bounds for c_i, $i \in [7]$ and $\sum_{i \in [7]} a_i$, which implies a lower bound on x. Then, by removing and adding weights between some vertices of weights a_i, a_j and b_{ij}, we get an upper bound on x, which gives a contradiction. The proofs for $k = 8$ and $k \geq 9$ are based on analogous ideas. The main differences come from the fact that the aforementioned bounds on c_i and x are derived using induction and the values of $f(k-2)$ and $f(k)$, which are distinct for each $k \geq 7$.

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