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A lower bound for Set-colouring Ramsey numbers

(Extended abstract)

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Abstract

The set-colouring Ramsey number $R_{r,s}(k)$ is defined to be the minimum n such that if each edge of the complete graph K_n is assigned a set of s colours from $\{1, \ldots, r\}$, then one of the colours contains a monochromatic clique of size k. The case $s = 1$ is the usual r-colour Ramsey number, and the case $s = r - 1$ was studied by Erdős, Hajnal and Rado in 1965, and by Erdős and Szemerédi in 1972.

The first significant results for general s were obtained only recently, by Conlon, Fox, He, Mubayi, Suk and Verstraëte, who showed that $R_{r,s}(k) = 2^{\Theta(kr)}$ if s/r is bounded away from 0 and 1. In the range $s = r - o(r)$, however, their upper and lower bounds diverge significantly. In this note we introduce a new (random) colouring, and use it to determine $R_{r,s}(k)$ up to polylogarithmic factors in the exponent for essentially all r, s and k.

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1 Introduction

The r-colour Ramsey number $R_r(k)$ is defined to be the minimum $n \in \mathbb{N}$ such that every r-colouring $\chi\colon E(K_n) \to \{1,\dots,r\}$ of the edges of the complete graph on n vertices contains a monochromatic clique of size k. These numbers (and their extensions to general graphs, hypergraphs, etc.) are among the most important and extensively-studied objects in combinatorics, see for example the beautiful survey article [4].

In this paper we will study the following generalisation of the r-colour Ramsey numbers.

Definition 1.1. The set-colouring Ramsey number $R_{r,s}(k)$ is the least $n \in \mathbb{N}$ such that every colouring $\chi\colon E(K_n) \to \binom{[r]}{s}$ contains a monochromatic clique of size k, that is, a set $S \subset V(K_n)$ with $|S| = k$ and a colour $i \in [r]$ such that $i \in \chi(e)$ for every $e \in \binom{S}{2}$.

That is, we assign a set $\chi(e) \subset [r] = \{1,\dots,r\}$ of s colours to each edge of the complete graph, and say that a clique is monochromatic if there exists a colour $i \in [r]$ that is assigned to every edge of the clique. Note that when $s = 1$ this is simply the usual r-colour Ramsey number. The study of set-colouring Ramsey numbers was initiated in the 1960s by Erdős, Hajnal and Rado [5], who conjectured that $R_{r,r-1}(k) \le 2^{\delta(r)k}$ for some function $\delta(r) \to 0$ as $r \to \infty$. This conjecture was proved by Erdős and Szemerédi [6] in 1972, who showed that

$$2^{\Omega(k/r)} \le R_{r,r-1}(k) \le r^{O(k/r)}.$$

For more general values of s, the first significant progress was made only recently, by Conlon, Fox, He, Mubayi, Suk and Verstraëte [2], who showed that

$$\exp\left(\frac{c'k(r-s)^3}{r^2}\right) \le R_{r,s}(k) \le \exp\left(\frac{ck(r-s)^2}{r}\log\frac{r}{\min\{s,r-s\}}\right) \tag{1}$$

for absolute constants $c, c' > 0$. While the exponents in the lower and upper bounds differ by only a factor of $\log r$ when $r - s = \Omega(r)$, they diverge much more significantly when $(r-s)/r \to 0$. We remark that the range $s = r - o(r)$ was of particular interest to the authors of [2], who were motivated by an application to hypergraph Ramsey numbers [3].

The main result of this paper is the following improved lower bound, which allows us to determine $R_{r,s}(k)$ up to a poly-logarithmic factor in the exponent for essentially all r, s, k.

Theorem 1.2. *There exist constants $C > 0$ and $\delta > 0$ such that the following holds. If $r, s \in \mathbb{N}$ with $s \le r - C\log r$, then*

$$R_{r,s}(k) \ge \exp\left(\frac{\delta k(r-s)^2}{r}\right) \tag{2}$$

for every $k \ge (C/\varepsilon)\log r$, where $\varepsilon = (r-s)/r$.

Note that the bound (2) matches the upper bound (1) on $R_{r,s}(k)$, proved in [2], up to a factor of $O(\log r)$ in the exponent for all $s \le r - C\log r$. When $s \ge r - C\log r$ our method does not provide a construction, but in this case the bounds from [2] only differ by a factor of order $(\log r)^2$ in the exponent, the lower bound coming from a simple random colouring.

The lower bound on k in Theorem 1.2 is also not far from best possible, since if $k \le 1/\varepsilon$ then the most common colour has density at least $1 - 1/k$, and therefore $R_{r,s}(k) \le k^2$, by Turán's theorem. A simpler version of the construction described in this paper (taking complete $(k-1)$-partite graphs instead of blow-ups of random graphs) extends Theorem 1.2 to a wider range of s and k, as stated in the following corollary. We omit its proof for space reasons.

Corollary 1.3. *Let $r > s \ge 1$ and $\delta > 0$, and set $\varepsilon = (r-s)/r$. We have $R_{r,s}(k) = 2^{\tilde{\Theta}(\varepsilon^2 rk)}$ for every $k \ge (1+\delta)/\varepsilon + 1$.*

2 The construction

In this section we will define the (random) colouring that we use to prove Theorem 1.2, and prove that it has the desired properties with high probability. The idea behind our construction, to let each colour be a random copy of some pseudorandom graph, was introduced in the groundbreaking work of Alon and Rödl [1] on multicolour Ramsey numbers, and has been used in several recent papers in the area [7, 8, 10, 9]. However, our approach differs from that used in these previous works in several important ways; in particular, we will not count independent sets, and it will be important that our colour classes are chosen (almost) independently at random.

Fix a sufficiently small[1] constant $\delta > 0$, and set $C = 1/\delta^3$. Recall that $r - s = \varepsilon r$, and let

$$m = 2^{\delta^2 \varepsilon k} \qquad \text{and} \qquad n = 2^{\delta^4 \varepsilon^2 rk}.$$

Note that $\varepsilon\sqrt{m} \ge k$, since $k \ge (C/\varepsilon)\log r$ and $\varepsilon \ge 1/r$, and by our choice of C.

Set $p = 1 - 5\delta\varepsilon$, and for each colour $i \in [r]$, let

- H_i be an independently chosen copy of the random graph $G(m,p)$, and
- $\phi_i \colon [n] \to [m]$ be an independently and uniformly chosen random function.

Now define G_i to be the (random) graph with vertex set $[n]$ and edge set

$$E(G_i) = \big\{ uv : \{\phi_i(u), \phi_i(v)\} \in E(H_i) \big\},$$

that is, a random blow-up of H_i, with parts given by ϕ_i. Define a colouring χ' of K_n by $\chi'(e) = \big\{ i \in [r] : e \in E(G_i) \big\}$, and define the set of *bad* edges to be

$$B = \big\{ e \in E(K_n) : |\chi'(e)| < s \big\}. \tag{3}$$

We will also say that an edge $e = uv \in E(K_n)$ is *i-crossing* if $\phi_i(u) \ne \phi_i(v)$, and define

$$\kappa(e) = \big\{ i \in [r] : e \text{ is } i\text{-crossing} \big\}.$$

We can now define the colouring that we will use to prove Theorem 1.2.

[1] In fact taking $\delta = 2^{-5}$ would suffice, but we will not make any attempt to optimise the value of δ.

Definition 2.1. For each $e \in E(K_n)$, we define the set of colours $\chi(e) \subset [r]$ by

$$\chi(e) = \begin{cases} \chi'(e) & \text{if} \quad e \notin B, \\ \kappa(e) & \text{if} \quad e \in B. \end{cases}$$

Our task is to show that with high probability $|\chi(e)| \ge s$ for every $e \in E(K_n)$, and moreover that χ contains no monochromatic copy of K_k. We start with the former.

Lemma 2.2. *With high probability, $|\chi(e)| \ge s$ for every $e \in E(K_n)$.*

Proof. Note that for each $i \in [r]$ we have $\Pr(i \notin \kappa(e)) = 1/m$ all independently, by the definition of the functions ϕ_i. By the union bound over the set of $r - s = \varepsilon r$ missed colours,

$$\Pr\big(|\chi(e)| < s\big) \le \Pr\big(|\kappa(e)| < s\big) \le \binom{r}{\varepsilon r}\left(\frac{1}{m}\right)^{\varepsilon r} \le \left(\frac{e}{\varepsilon m}\right)^{\varepsilon r} \le 2^{-\delta^3\varepsilon^2 rk} \le \frac{1}{n^3}$$

since $k \ge (C/\varepsilon)\log r$ and $C = \delta^{-3}$ imply that $\varepsilon m \ge \sqrt{m} = 2^{\delta^2\varepsilon k/2}$. Applying Markov's inequality and taking an union bound over edges then proves the lemma. □

To prove that χ contains no monochromatic copy of K_k, we split into two cases, the easier case being the following. Let $t = \delta\varepsilon k^2$.

Lemma 2.3. *With high probability, the colouring χ contains no monochromatic k-clique with at most t bad edges.*

Proof. Suppose χ contains a monochromatic clique $S = \{v_1, \dots, v_k\}$ of colour $i \in [r]$ such that at most t of the edges $e \in \binom{S}{2}$ are bad. For each $j \in [k]$, let $w_j = \phi_i(v_j) \in V(H_i)$, and observe that the set $W = \{w_1, \dots, w_k\}$ has size k, since by Definition 2.1, and noting that $\chi'(e) \subset \kappa(e)$, every edge $e \in E(K_n)$ such that $i \in \chi(e)$ is i-crossing.

Now, if $e = v_j v_\ell \in \binom{S}{2}$ is not a bad edge, then $i \in \chi(e) = \chi'(e)$, and hence $w_j w_\ell \in E(H_i)$. Since there are at most t bad edges in $\binom{S}{2}$, it follows that $e\big(H_i[W]\big) \ge \binom{k}{2} - t > p\binom{k}{2} + \delta\varepsilon k^2$, since $p = 1 - 5\delta\varepsilon$ and $t = \delta\varepsilon k^2$. Since $H_i[W] \sim G(k,p)$, it follows from Chernoff's inequality that this event has probability at most $e^{-\delta^2\varepsilon k^2}$. By the union bound, the probability that χ contains a monochromatic clique with at most t bad edges is at most

$$r\binom{m}{k}e^{-\delta^2\varepsilon k^2} \le r\big(2^{\delta^2\varepsilon k} \cdot e^{-\delta^2\varepsilon k}\big)^k.$$

Since $\delta^3\varepsilon k \ge \log r$, the right-hand side tends to zero as $k \to \infty$, as required. □

Our remaining task is to show that, with high probability, no graph F of the family

$$\mathcal{F} = \big\{F \subset K_n \,:\, v(F) = k \ \text{ and } \ e(F) = t\big\}$$

is such that $F \subset B$.[2] We will not be able to prove this using a simple first-moment argument, summing over all graphs $F \in \mathcal{F}$, since the probability of the event $\{F \subset B\}$ is not always sufficiently small. Instead, we will identify a 'bottleneck event' for each $F \in \mathcal{F}$.

[2]Here, and below, we abuse notation slightly by treating the set of bad edges B as a graph.

To do so, choose a total ordering $\prec_F$ on the vertices of F such that $u \prec_F v$ implies $d_F(u) \ge d_F(v)$. In other words, we order the vertices according to their degrees in F, breaking ties arbitrarily. Now, define

$$Q_i(F) = \big\{ v \in V(F) \,:\, \exists\, u \in V(F) \text{ with } u \prec_F v \text{ such that } \phi_i(u) = \phi_i(v) \big\}$$

to be the set of vertices which share a part of ϕ_i with another vertex of F that comes earlier in the order $\prec_F$. We remark that if $u \prec_F v$, then $u \neq v$. In the following two lemmas, we will bound the probability in two different ways, depending on the size of

$$X_F = \sum_{i=1}^{r} \sum_{v \in Q_i(F)} d_F(v).$$

Lemma 2.4. *With high probability, there does not exist $F \in \mathcal{F}$ with $X_F \le \varepsilon rt/2$ and $F \subset B$.*

Proof. We first reveal the random functions $\phi_1, \dots, \phi_r$, and therefore the sets $Q_i(F)$ (and hence also the random variable X_F) for each $F \in \mathcal{F}$. To prove the lemma we will only use the randomness in the choice of $H_1, \dots, H_r$. More precisely, we will consider the set

$$Y = \big\{ (uv, i) \in E(F) \times [r] : u, v \notin Q_i(F) \big\}$$

of pairs $(e, i) \in E(F) \times [r]$ such that neither endpoint of e is contained in $Q_i(F)$, and

$$Z = \sum_{(e,i) \in Y} \mathbb{1}\big[e \notin E(G_i) \big],$$

the number of such pairs for which $i \notin \chi'(e)$. Note that, for each $i \in [r]$, the graph $\{e : (e, i) \in Y\}$ is contained in a clique with at most one vertex in each part of ϕ_i. The events $\{e \in E(G_i)\}$ for $(e, i) \in Y$ are therefore independent, and hence $Z \sim \mathrm{Bin}(|Y|, 1-p)$. Since $|Y| \le rt$, Z is dominated by a binomial random variable with expectation $(1-p)rt = 5\delta\varepsilon rt$.

If $F \subset B$, then for each edge $e \in E(F)$, there are at least εr colours $i \in [r]$ such that $e \notin E(G_i)$. Thus

$$\sum_{i=1}^{r} \sum_{e \in E(F)} \mathbb{1}\big[e \notin E(G_i) \big] \ge \varepsilon rt.$$

Therefore, if $X_F \le \varepsilon rt/2$, then $Z \ge \varepsilon rt/2$, since for each vertex $v \in Q_i(F)$ we remove at most $d_F(v)$ edges from Y. By Chernoff's inequality, it follows that for a fixed $F \in \mathcal{F}$ we have $\Pr(X_F \le \varepsilon rt/2 \text{ and } F \subset B) \le e^{-\delta\varepsilon rt}$. Taking a union bound and recalling that $t = \delta\varepsilon k^2$, it follows that the probability that there exists $F \in \mathcal{F}$ with $X_F \le \varepsilon rt/2$ and $F \subset B$ is at most

$$\binom{n}{k} \binom{\binom{k}{2}}{t} e^{-\delta\varepsilon rt} \le \left(\frac{en}{k} \left(\frac{e}{\delta\varepsilon} \right)^{\delta\varepsilon k} e^{-\delta^2\varepsilon^2 rk} \right)^{k} \to 0,$$

as claimed, where in the final step we used our choice of $n = 2^{\delta^4\varepsilon^2 rk}$, the bound $\varepsilon \ge (\log r)/r$, which holds by our assumption that $s \le r - C\log r$, and our choice of $C = 1/\delta^3$. $\square$

Finally, we will use the randomness in $\phi_1, \dots, \phi_r$ to show that X_F is always small.

Lemma 2.5. *With high probability, $X_F \le \varepsilon rt/2$ for every $F \in \mathcal{F}$.*

Proof. For each graph $F \in \mathcal{F}$, and each $j \in \big\{1, \dots, \lceil \log_2 k \rceil\big\}$, define

$$A_j(F) = \Big\{ v \in V(F) : 2^{-j}k \le d_F(v) < 2^{-j+1}k \Big\} \quad \text{and} \quad s_j(F) = \sum_{i=1}^{r} |A_j(F) \cap Q_i(F)|.$$

Note that the random functions $\phi_1, \dots, \phi_r$ determine $Q_1(F), \dots, Q_r(F)$, and hence $s_j(F)$. The key step is the following claim, which provides us with our bottleneck event.

Claim. *If $X_F \ge \varepsilon rt/2$, then there exists $\ell \in \big\{1, \dots, \lceil \log_2 k \rceil\big\}$ satisfying the inequality $s_\ell(F) > \delta \varepsilon r \sum_{j=1}^{\ell} |A_j(F)|$.*

Proof of claim. Observe that

$$\frac{\varepsilon rt}{2} \le X_F = \sum_{i=1}^{r} \sum_{v \in Q_i(F)} d_F(v) \le \sum_{i=1}^{r} \sum_{j=1}^{\lceil \log_2 k \rceil} \frac{k}{2^{j-1}} \cdot |A_j(F) \cap Q_i(F)| = \sum_{j=1}^{\lceil \log_2 k \rceil} \frac{k}{2^{j-1}} \cdot s_j(F).$$

Thus, if the conclusion of the claim fails to hold for every $\ell \in \big\{1, \dots, \lceil \log_2 k \rceil\big\}$ then we have

$$\begin{aligned} \frac{t}{4\delta k} &\le \frac{1}{\delta \varepsilon r} \sum_{\ell=1}^{\lceil \log_2 k \rceil} \frac{s_\ell(F)}{2^\ell} \le \sum_{\ell=1}^{\lceil \log_2 k \rceil} \frac{1}{2^\ell} \sum_{j=1}^{\ell} |A_j(F)| \\ &= \sum_{j=1}^{\lceil \log_2 k \rceil} |A_j(F)| \sum_{\ell=j}^{\lceil \log_2 k \rceil} \frac{1}{2^\ell} \le \sum_{j=1}^{\lceil \log_2 k \rceil} \frac{|A_j(F)|}{2^{j-1}} \le \frac{2}{k} \sum_{v \in V(F)} d_F(v) = \frac{4t}{k}. \end{aligned}$$

Since $\delta < 2^{-4}$, this is a contradiction, and so the claim follows. □

Fix $\ell \in \big\{1, \dots, \lceil \log_2 k \rceil\big\}$ such that the conclusion of the claim holds, and set $A := \cup_{j=1}^{\ell} A_j(F)$ and $a := |A|$. Now, if we reveal ϕ_i for the vertices of F one vertex at a time using the order $\prec_F$, then for each vertex $v \in Q_i(F)$ we must choose $\phi_i(v)$ to be one of the (at most k) previously selected elements of $[m]$. The expected number of sets A such that the conclusion of the claim holds is thus at most

$$\sum_{a=1}^{k} n^a \binom{ar}{\delta \varepsilon ar} \left(\frac{k}{m}\right)^{\delta \varepsilon ar} \le \sum_{a=1}^{k} \left(n \cdot \left(\frac{e}{\delta \varepsilon} \cdot \frac{k}{m} \right)^{\delta \varepsilon r} \right)^a \to 0$$

as $k \to \infty$, as required, since $n = 2^{\delta^4 \varepsilon^2 rk}$ and $\varepsilon m/k \ge \sqrt{m} = 2^{\delta^2 \varepsilon k/2}$. □

Proof of Theorem 1.2. Combining Lemmas 2.2, 2.3, 2.4 and 2.5, we see that, with high probability, the random colouring χ satisfies $|\chi(e)| \ge s$ for every $e \in E(K_n)$ and contains no monochromatic K_k. Therefore $R_{r,s}(k) > n = 2^{\delta^4 \varepsilon^2 rk}$, as desired. □

References

[1] N. Alon and V. Rödl, Sharp bounds for some multicolor Ramsey numbers, *Combinatorica*, **25** (2005), 125–141.

[2] D. Conlon, J. Fox, X. He, D. Mubayi, A. Suk and J. Verstraëte, Set-coloring Ramsey numbers via codes, arXiv:2206.11371

[3] D. Conlon, J. Fox, X. He, D. Mubayi, A. Suk and J. Verstraëte, Hypergraph Ramsey numbers of cliques versus stars, *Random Structures Algorithms*, to appear, arXiv:2210.03545

[4] D. Conlon, J. Fox and B. Sudakov, Recent developments in graph Ramsey theory, Surveys in Combinatorics, **424** (2015), 49–118.

[5] P. Erdős, A. Hajnal and R. Rado, Partition relations for cardinal numbers, *Acta Math. Acad. Sci.*, **16** (1965), 93–196.

[6] P. Erdős and A. Szemerédi, On a Ramsey type theorem, *Period Math. Acad.*, **2** (1972), 295–299.

[7] X. He and Y. Wigderson, Multicolor Ramsey numbers via pseudorandom graphs, *Electron. J. Combin.*, **27** (2020), Article No. P1.32.

[8] D. Mubayi and J. Verstraëte, A note on pseudorandom Ramsey graphs, *J. Eur. Math. Soc.*, to appear, arXiv:1909.01461.

[9] W. Sawin, An improved lower bound for multicolor Ramsey numbers and a problem of Erdős, *J. Combin. Theory Ser. A*, **188** (2022), Paper No. 105579, 11 pp.

[10] Y. Wigderson, An improved lower bound on multicolor Ramsey numbers, *Proc. Amer. Math. Soc.*, **149** (2021), 2371–2374.