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Moderate Deviations of Triangle Counts – the lower tail

(Extended abstract)

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Abstract

Two recent papers [11] and [19] study the lower tail of triangle count deviations in random graphs $G(n,m)$ with positive density $t := m/\binom{n}{2} \in (0,1)$. Let us write $D_\triangle(G)$ for the deviation of the triangle count from its mean. Results of [11] and [19] determine the order of magnitude of the log probability $\log(\mathbb{P}\left(D_\triangle(G(n,m)) < -\tau\binom{n}{3}\right))$ for the ranges $n^{-3/2} \ll \tau \ll n^{-1}$ and $n^{-3/4} \ll \tau \ll 1$ respectively. Furthermore, in [19] it is proved that the log probability is at least $\Omega(\tau^2 n^3)$ in the "missing" range $n^{-1} \ll \tau \ll n^{-3/4}$, and they conjectured that this result gives the correct order of magnitude. Our main contribution is to prove this conjecture.

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1 Introduction

The study of subgraph count deviations, and especially triangle count deviations has been a very active area of research in recent decades. In particular, a great many results have been proved regarding small deviations (of the order of the standard deviation) beginning

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with Ruciński [22], see also [2, 14, 13, 15, 17, 20, 21]. There have also been many results which focus on large deviations (of the order of the mean) including the seminal articles of Vu [23] and Janson and Ruciński [16] in the early 2000s, and continuing with Chatterjee and Varadhan [4] who related these deviations to solutions to variational problems, which were resolved in certain cases by Lubetzky and Zhao [18] and Zhao [24]. The survey of Chatterjee [3] and the references therein give a detailed overview. Further developments related to these techniques may be found in [1, 5, 9]. A major breakthrough by Harel, Mousset and Samotij [12] essentially resolved the large deviation (upper tail) problem for triangles.

There has also been some interest in deviations of intermediate value, which we call moderate deviations. These deviations are considered in the $G(n,p)$ model in [7, 8, 10]. It is argued by the third author, together with Goldschmidt and Scott [11] that for many moderate deviation problems the $G(n,m)$ model is more appropriate as it is possible to study finer causes of deviatons, and that, in any case, one may deduce results for $G(n,p)$ by a simple conditioning argument. See also [6], which extends these results to sparser random graphs.

Let us now consider the model $G_m \sim G(n,m)$, in which G_m is selected uniformly from graphs with n vertices and m edges. Suppose that $t \in (0,1)$ is fixed and that our random graphs have density t, that is $t = m/\binom{n}{2}$. Let $N_\triangle(G)$ be the number of triangles in the graph G and let $D_\triangle(G_m)$ be the deviation of the triangle count in G_m, i.e., $D_\triangle(G_m) := N_\triangle(G_m) - \mathbb{E}\left[N_\triangle(G_m)\right]$.

We also remark that the majority of results previously mentioned have focussed on the upper tail, whereas we shall focus on the lower tail. That is, we consider the question of how likely it is that a random graph has many *fewer* triangles than expected.

By the main results of [11] and Neeman, Radin and Sadun [19] respectively, we have

$$-\log\left[\mathbb{P}\left(D_\triangle(G_m) < -\tau\binom{n}{3}\right)\right] = \begin{cases}\Theta(\tau^2 n^3) & n^{-3/2} \ll \tau \ll n^{-1}\\ \Theta(\tau^{2/3} n^2) & n^{-3/4} \ll \tau \ll 1\end{cases}$$

Furthermore, Neeman, Radin and Sadun [19] obtained the bounds

$$\exp(-C\tau^{2/3}n^2) \leqslant \mathbb{P}\left(D_\triangle(G_m) < -\tau\binom{n}{3}\right) \leqslant \exp(-c\tau^2 n^3)$$

in the “missing” range $n^{-1} \ll \tau \ll n^{-3/4}$, for some constants $c, C > 0$. They conjectured that the final quantity is the correct probability of this deviation, up to the choice of the constant C. We prove this conjecture, thus completing the understanding of the order of magnitude of deviations in the lower tail across essentially the entire range of possible deviations.

Theorem 1. *Let $t \in (0,1)$. There exists a constant $c > 0$ such that the following holds. Suppose that n is sufficiently large and $c^{-1}n^{-1} \leqslant \tau \leqslant cn^{-3/4}$ then*

$$\mathbb{P}\left(D_\triangle(G_m) < -\tau\binom{n}{3}\right) \geqslant \exp(-c\tau^2 n^3)\,.$$

2 Preliminaries

As we are claiming a lower bound on the deviation probability we must justify that there is a certain reasonably likely "cause" of this deviation. In works that consider the upper tail this is often a fixed subgraph (such as a clique or hub) which occurs with a certain probability. As Neeman, Radin and Sadun [19] discovered, the situation is more subtle for the lower tail. In the range of slightly larger deviations they showed that the likeliest "cause" corresponds to a deviation event of the smallest eigenvalue.

We shall give a quite different "cause" of the triangle deficit. Roughly speaking, we consider running the majority of the process, and then, near the end, we select a certain set of pairs (non-edges) which have *small* codegree. If in the rest of the process we select many more of these pairs than expected then this causes a deficit of triangles in the final graph G_m. We show that this cause has a cost of $\exp\left(-\Theta(\tau^2 n^3)\right)$, thus proving Theorem 1.

In fact, to implement this approach we have to be slightly more careful about the set of pairs of low co-degree, which we will call F_-. It will be useful that F_- is close to regular. We therefore introduce a concept of synergy, which we use instead of codegree when defining F_-.

Notation

We write $d_u(G)$ for the degree of a vertex u in G and $d_{uv}(G)$ for the codegree of the pair u, v.

We now define *synergy*. The synergy of u and v with respect to G is

$$Syn_{uv}(G) := d_{uv}(G) - td_u(G) - td_v(G) + t^2(n-2).$$

The synergy of a pair of vertices can be thought of as how well their neighbourhoods intersect. As we are dealing with well behaved graphs, i.e. graphs with high probability properties, if a pair has positive synergy, then a high proportion of the neighbourhoods of its vertices intersect, and the opposite is true for negative synergy.

As we said, our proof involves revealing G_m into two parts, which we call G_m^0 and G_m^1. The first part will correspond to the first $m_0 := (1-\eta)m$ edges added to G_m, where $\eta \in (0,1)$. We note that $G_m^0 \sim G(n,(1-\eta)m)$, and we shall assume at various points in the proof that G_m^0 has the standard properties which hold with high probability in such random graphs.

The negative deviation of triangles will come with the selection of G_m^1. We note that G_m^1 corresponds to the last $m_1 := \eta m$ edges of the random process.

Let (f_i) be the sequence of non-edges of G_0 with non-decreasing order of synergies. The set of non-edges of low synergy is

$$F_- := \left\{ f_i : i \in \left\{ 1, \ldots, \frac{\binom{n}{2} - m_0}{2} \right\} \right\}$$

and the set of non-edges of high synergy is

$$F_+ := \left\{ f_i : i \in \left\{ \frac{\binom{n}{2} - m_0}{2} + 1, \ldots, \binom{n}{2} - m_0 \right\} \right\}.$$

Discussion of our approach

Let us first ask: What is the expected value of $|F_- \cap E(G_m^1)|$? Since F_- and F_+ have the same size, we have that $\mathbb{E}\left[|F_- \cap E(G_m^1)|\right] = \mathbb{E}\left[|F_+ \cap E(G_m^1)|\right] = \frac{m_1}{2} = \frac{\eta}{2}m$. Moreover, the same equalities holds if we replace the expectation by the conditional expectation (given G_m^0). With this in mind, we define the event

Definition 2 (The Event $\mathcal{E}(\alpha)$)**.** Let $\alpha \in (0,1)$ be a parameter. We denote by $\mathcal{E}(\alpha)$ the event defined by "$|F_- \cap E(G_m^1)| = (1+\alpha)m_1/2$".

Note that this event relies on first revealing G_m^0, as F_- is defined as a function of G_m^0. Since pairs of low synergy tend to have smaller codegree, there ought to be a relation between selecting more edges of G_m^1 in F_- and a deficit of triangles in the final random graph G_m. This idea is central to our approach.

3 Main result

As we will simply provide a proof overview here, some of the details will be left somewhat vague. For example, it is useful to have a graph property $\mathcal{P}_0$ such that $\mathbb{P}_{G_0}(\mathcal{P}_0) = 1 + o(1)$. This graph property consist of various properties which hold with high probability in random graphs. We note that, by monotony and conditioning, and the fact that $\mathcal{E}(\alpha)$ is independent of G_m^0 (and so also $\mathcal{P}_0$) we have

$$\begin{aligned}\mathbb{P}\left(N_\triangle(G) < \mathbb{E}\left[N_\triangle(G)\right] - a\right) \geqslant \\ (1+o(1))\,\mathbb{P}\left(\mathcal{E}(\alpha)\right)\,\mathbb{P}\big(N_\triangle(G) < \mathbb{E}\left[N_\triangle(G)\right] - a \,\big|\, \mathcal{E}(\alpha), \mathcal{P}_0\big).\end{aligned}$$

Given this inequality, it suffices to prove the following two lemmas:

Lemma 3. *Let $\alpha := \alpha_n$ with $n^{-1} \ll \alpha_n \ll n^{-1/4}$. Then*

$$\mathbb{P}\left(\mathcal{E}(\alpha)\right) \geqslant \exp\left(-O_{t,\eta}(\alpha^2 n^2)\right). \tag{3.1}$$

Lemma 4. *There exists $C > 0$ such that the following holds. If n is sufficiently large and $\alpha n^{5/2} \geqslant Ca$ then*

$$\mathbb{P}\big(N_\triangle(G) < \mathbb{E}\left[N_\triangle(G)\right] - a \,\big|\, \mathcal{E}(\alpha), \mathcal{P}_0\big) = n^{-O_{t,\eta}(1)} = e^{o(n)}. \tag{3.2}$$

We remark that Lemma 3 follows easily from known bounds on the tail of the hypergeometric distribution. The proof of Lemma 4 is more involved. However, the result clearly follows from the following statement (and Markov's inequality)

$$\mathbb{E}\left[N_{\triangle}(G) \mid \mathcal{E}(\alpha), \mathcal{P}_0\right] \leqslant \mathbb{E}\left[N_{\triangle}(G)\right] - 2a\,. \tag{3.3}$$

To prove (3.3) we consider the various types of triangle which occur in the final graph. We divide the triangle count into four categories: three edges from G_m^0, $\triangle_{(3,0)}$, two edges from G_m^0 and one from G_m^1, $\triangle_{(2,1)}$, one edges from G_m^0 and two from G_m^1, $\triangle_{(1,2)}$ and three edges from G_m^1, $\triangle_{(0,3)}$. The idea is that the number of edges of type $(3,0)$ is predictable, as G_m^0 is a random graph; the number of type $(2,1)$ is significantly *less* than one would expect, because we are conditioning on the event $\mathcal{E}(\alpha)$; and we shall prove that the conditioning does not change the expected number of types $(1,2)$ and $(0,3)$ by very much.

The following result states that conditioning on $\mathcal{E}(\alpha)$ does indeed have the effect of reducing the expected number of triangles of type $(2,1)$. The result mentions μ_- and μ_+ which are defined to be the average codegree of pairs in F_- and F_+ respectively. One of the properties in $\mathcal{P}_0$ is that $\mu_+ - \mu_- = \Omega(n^{1/2})$.

Lemma 5.

$$\mathbb{E}\left[\triangle_{(2,1)}|\mathcal{P}_0, \mathcal{E}_\eta(\alpha)\right] = \mathbb{E}_{G_m^0}[\triangle_{(2,1)}] + \alpha m \eta(\mu_- - \mu_+)$$

The alert reader may question why we chose to define F_- in terms of synergy, rather than simply taking F_- to be those pairs with smaller codegree. Indeed, Lemma 5 would work just as well with this alternative definition. However, the problem arises when trying to control the effect that conditioning (on $\mathcal{E}(\alpha)$) has on the expected number of triangles of types $(1,2)$ and $(0,3)$. We are able to prove sufficiently strong bounds

$$\begin{aligned} \mathbb{E}\left[\triangle_{(1,2)}|\mathcal{P}_0, \mathcal{E}_\eta(\alpha)\right] &= O(\alpha\eta m n^{1/2}) \quad \text{and} \\ \mathbb{E}\left[\triangle_{(0,3)}|\mathcal{P}_0, \mathcal{E}_\eta(\alpha)\right] &= O(\alpha\eta m n^{1/2}) \end{aligned}$$

using the fact that F_- is close to regular. (These bounds are seen to be sufficient by taking the constant η sufficiently small.) We remark that our proofs of these bounds rely on the fact that F_- is close to regular, and so would fail if F_- was defined directly in terms of codegrees.

In order to prove F_- is close to regular we actually prove the following stronger statement about the set of synergies $Syn_{uw}(G_m^0) : w \in V \setminus N_u$ between a fixed vertex u and its non-neighbours w. Let σ denote the standard deviation of $Syn_{uw}(G_m^0)$ (which is of order $n^{1/2}$). The following result states an approximate central limit theorem for the empirical distribution of synergies.

Lemma 6. *There exists a constant $C > 0$ such that, with high probability the following holds simultaneously for all vertices $u \in V(G_m^0)$:*

$$\left|\{w \in V \setminus N_u \,:\, Syn_{uw}(G_m^0) \leqslant a\sigma\}\right| = \left(\Phi(a) \pm Cn^{-1/4}\right)(n - d_u(G_m^0))\,.$$

Using this approximation of the distribution of synergies together with other concentration bounds and tools such as Goodman's theorem it is possible to control the expected number of triangles of type $(1,2)$ and $(0,3)$ and thereby prove (3.3).

4 Remarks

As we said, Neeman, Radin and Sadun [19] showed that their construction for the lower bound holded even for the missing range. Their construction involved partitioning the graph into two parts in which the smaller part (much smaller than the other) have a lower density of edges. Hence, they focused on partitioning the vertex and we, instead, partitioned the edges, which should be more effective for $G(n, m)$ type of graphs since their structure is more rigid.

There are a number of questions which remain open. For example, is it possible to extend these results, and the results of [19] to sparser random graphs, as [6] did with the results of [11]. One may also ask whether stronger bounds may be proved. Perhaps it is possible to determine the log probability asymptotically, rather than up to a multiplicative constant. It seems possible to divide our construction further into finer steps with scaling tendencies towards low synergy pairs to get optimal results.

Finally, it would be interesting to investigate other graphs. We remark that the results we prove here correspond to a regime which simply doesn't exist for odd cycles of length at least 5. Surprisingly [19] showed that the log probability exhibits a large discontinuity when considering odd cycles of length at least 5.

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