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Partition universality for hypergraphs of bounded degeneracy and degree

(Extended abstract)

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Abstract

We consider the following question. When is the random k-uniform hypergraph $\Gamma = G^{(k)}(N,p)$ likely to be r-partition universal for k-uniform hypergraphs of bounded degree and degeneracy? That is, for which p can we guarantee asymptotically almost surely that in any r-colouring of $E(\Gamma)$ there exists a colour χ such that in Γ there are χ-monochromatic copies of all k-uniform hypergraphs of maximum vertex degree Δ, degeneracy at most D, and cN vertices for some constant $c = c(D,\Delta) > 0$. We show that if $\mu > 0$ is fixed, then $p \geq N^{-1/D+\mu}$ suffices for a positive answer if N is large. On the other hand, for $p = o(N^{-1/D})$ we show that $G^{(k)}(N,p)$ is likely not to contain some graphs of maximum degree Δ and degeneracy D on cN vertices at all.

This improves the best upper bounds on the minimum number of edges required for a k-uniform hypergraph to be partition universal (even for $k = 2$) and also for the size-Ramsey problem for most k-uniform hypergraphs of bounded degree and degeneracy.

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1 Introduction

Ramsey theory refers to the area of combinatorics that studies unavoidable organised subsystems. At the centre of this area of research is Ramsey's [24] seminal result which states

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that for any graph F there exists N such that in any 2-edge-colouring of K_N there is a monochromatic copy of F. This is often indicated with the notation $K_N \to_2 F$. The study of the minimal value of N for which $K_N \to_2 F$, which is denoted by $R_2(F)$ and called the *Ramsey number* of F, is one of the main topics of Ramsey theory. This question can be easily generalised to a setting with r colours and a different host graph G: we write $G \to_r F$ if every r-colouring of $E(G)$ contains a monochromatic copy of F.

Another line of research that generated a lot of attention was started by Erdős, Faudree, Rousseau and Schelp [13]. The authors analysed other sparser structures for which we have $G \to_2 F$ and in particular, for fixed F and $r \in \mathbb{N}$, they asked to determine what they called the *size-Ramsey number* of the graph F, defined as $\hat{R}_r(F) := \min\{|E(G)| : G \to_r F\}$. We remark that this well defined as $K_{R_r(F)} \to_r F$.

One can also ask for a stronger property of G, namely that it is r-partition-universal for a family $\mathcal{F}$ of graphs. We say that G is r-partition-universal for $\mathcal{F}$ if for every r-colouring of $E(G)$, there exists some colour χ such that a colour χ copy of every $F \in \mathcal{F}$ appears. Observe that while $G \to_r F$ for every $F \in \mathcal{F}$ is certainly necessary for r-partition universality, it is not sufficient: it could be that for an r-colouring of $E(G)$ different members of $\mathcal{F}$ appear in different colours.

These questions have primarily been studied for graphs, but the above definitions naturally extend to k-uniform hypergraphs with $k \geq 3$. We will write *k-graphs* for k-uniform hypergraphs.

2 Previous results

For several classes of 'tree-like' graphs, the size-Ramsey number is known to be linear in the number of vertices. Beck [5] proved that $\hat{R}_r(P_n)$ is linear in n. Trees were dealt with by Friedman and Pippenger [14], and Haxell, Kohayakawa and Łuczak [17] proved it for cycles. Much later, Clemens et al. [7] showed that the same holds for powers of paths with 2 colours (Han et al. [15] extended this to r colours), Berger et al. [6] and Kamčev, Liebenau, Wood, and Yepremyan[18] for graphs of bounded maximum degree and treewidth.

For k-graphs for $k \geq 3$, Han et al. [16] proved a linear bound for 3-uniform tight paths, and Letzter, Pokrovskiy, Yepremyan [22] for all uniformities and more generally powers of bounded degree hypertrees.

However, not all bounded degree graphs have linear size-Ramsey numbers. Rödl and Szemerédi [25] proved that there is a family of graphs of maximum degree 3 whose size-Ramsey numbers grow as $n \log^{1/60} n$. Recently, Tikhomirov [27] improved this to $n \exp\big(\Omega(\sqrt{\log n})\big)$. In k-graphs, Dudek, Fleur, Mubayi and Rödl [12] proved a lower bound similar to that of Rödl and Szemerédi for k-graphs of maximum degree $k+1$. It remains a conjecture that these bounds can be improved to $n^{1+\varepsilon}$ (possibly for larger maximum degrees).

In terms of general upper bounds, Kohayakawa, Rödl, Schacht and Szemerédi [21] proved the first non-trivial upper bound $O\big(n^{2-1/\Delta} \log^{1/\Delta} n\big)$ on the size-Ramsey number of any graph on n vertices and maximum degree Δ; this was recently improved by Draganić and Petrova [11] for $\Delta = 3$.

For k-graphs, the upper bound $O(n^k)$ follows from the linearity of the usual Ramsey number, proved by Cooley, Fountoulakis, Kühn and Osthus [10]. The only improvement over this, to $O(n^{k-\varepsilon})$ for some (very) small $\varepsilon > 0$, was by Allen et al. [2].

All of the mentioned general upper bounds in fact are not only upper bounds on size-Ramsey numbers, but actually give graphs which are r-partition universal for bounded degree k-graphs.

3 Our result

If F is a k-graph, we write $\Delta(F)$ for the *maximum vertex degree* of F, i.e. the maximum number of edges which all contain a given vertex. We say F is D-degenerate if there is an ordering of the vertex set of F such that each vertex v in the ordering is in at most D edges whose other vertices are strictly before v. We prove the following main theorem.

Theorem 1. *For all $r, D, \Delta \in \mathbb{N}$, $k \geq 2$, and $\mu > 0$, there is $c > 0$ such that for all positive integers N and $p = N^{-\frac{1}{D}+\mu}$, the random k-graph $\Gamma = G^{(k)}(N, p)$ asymptotically almost surely has the following property. For every r-edge-colouring there is a colour χ such that, for any D-degenerate k-graph F of maximum degree at most Δ with cN vertices, there is a colour χ-monochromatic copy of F in Γ.*

For $k = 2$ this appears in the preprint [1, Theorem 3] of the first two authors.

Theorem 1 is asymptotically sharp. Indeed, if F is a D-degenerate graph on cN vertices with bounded maximum degree and approximately DcN edges (it is easy to construct such graphs) a first moment method argument shows that $G^{(k)}(N, p)$ with $p = o(N^{\frac{1}{D}})$ is likely not to contain a copy of F (let alone if we adversarily colour the edges of G).

4 Outline of the proof of Theorem 1

We now sketch the proof of Theorem 1. Our argument is intricate. Hence, to simplify the discussion here, we look at the case $k = 2$ first (and thus naturally refer to the lemmas in [1]), and then outline the changes needed for $k \geq 3$. For lack of space, we do not state the lemmas required precisely here, but only give a general idea.

Given a typical $\Gamma = G(N, p)$ and an adversarial r-colouring of its edges, we begin by identifying the colour χ and a subgraph G of colour χ edges in Γ into which we will embed our graph F. This G will be a h_1-partite subgraph of Γ, where h_1 is a reasonably large constant (depending on D, Δ and μ). We require that G has a property 'few unpromising subgraphs' which we will explain shortly. The precise statement is [1, Lemma 19].

We now want to embed a fixed D-degenerate graph F on cN vertices of maximum degree at most Δ to G. A method of Nenadov [23], which we formulate as a lemma in [1, Lemma 21], reduces this problem to that of finding 'robust homomorphisms' from F to G. Roughly speaking, this means we need to find a way of constructing a sequence

$\psi_0, \psi_1, \ldots, \psi_{v(F)}$ of partial homomorphisms from F to G, each embedding one more vertex than the previous, taken in the degeneracy order on G. The key property this sequence must have is: at each step, the number of choices we have to embed the next vertex is comparable to the number of choices we would have to embed it in the random graph Γ. That is, if the next vertex y of F has $d^-(y)$ neighbours before it in the degeneracy order, we should have $\Omega\big(p^{d^-(y)}N\big)$ choices for our embedding of y. The advantage this gives us is that we no longer have to care, when we construct our embedding of F, about avoiding previously used vertices.

We now explain the property 'few unpromising subgraphs'. Before we begin embedding F, we assign its vertices to the parts of G, with the property that vertices of F which are somewhat close together in graph distance will be assigned to different parts of G. When we come to embed y, we have already embedded its $d^-(y)$ neighbours $N^-(y)$ preceding it. In order to construct robust homomorphisms, we need many choices for our embedding of y. That is, we need that these embedded neighbours have many G-common neighbours in the part of G to which y is assigned. So the first idea is the following: we look at copies of $F\big[N^-(y)\big]$ in G, and designate them as *promising* if they have sufficiently many common neighbours in the correct part of G, and *unpromising* otherwise. For technical reasons—to be explained shortly—we actually have to modify this slightly, looking at copies not of $F\big[N^-(y)\big]$ but of a supergraph $H'_0(y)$ which contains some additional vertices of F that are within a (small) distance ℓ_0 of y. The property *few unpromising subgraphs* that we require of G is then the following. For each of the (boundedly many) choices of H'_0 and parts to which the vertices of H'_0 might be assigned, at most a tiny fraction of the embeddings of H'_0 into G are unpromising.

Our embedding procedure will succeed if we are never forced to embed $H'_0(y)$ to an unpromising copy in G. To avoid this, we need to look ahead a (large) constant ℓ_1 number of steps. That is, when we embed a vertex x of F preceding y, we look at the graph distance from x to y. If it exceeds ℓ_1, we will not *cross off for* y any vertices to which we could embed x. If it is at most ℓ_1, we will examine each candidate vertex v for x and determine whether v is *dangerous*, i.e. assigning v to x makes it too hard in the future to avoid embedding $H'_0(y)$ to an unpromising copy. If it does, we will *cross off* v, which means we will not choose to embed x to v. What we need to argue is that we will not need to cross off many vertices at any given step.

To decide whether v is dangerous, we look at all the embeddings of $H'_1(y)$ (all the vertices at distance ℓ_1 or less from y that precede y) that are consistent with our current embedding ψ_{x-1}. Some of these embed $H'_0(y)$ to an unpromising copy, and we call these *unpromising extensions*. If the fraction of unpromising extensions which embed x to v is exceptionally large, then v is dangerous.

We can use a double-counting argument to show that for vertices at distance strictly less than ℓ_1 from y, there can only be few vertices which are dangerous. For vertices at distance exactly ℓ_1 from y, this argument fails. However we can show that the embeddings 'mix rapidly' and there will in fact be no dangerous vertices in this case. This is [1, Lemma 20].

To complete this sketch, we require that before we begin to embed F, there are very few unpromising extensions of $H'_1(y)$. When $H'_0(y)$ is carefully chosen—this is the technical reason mentioned above—this follows from a theorem of Spencer [26] counting rooted copies of graphs in a random graph together with the few unpromising subgraphs property of G.

Putting the pieces together, when we use this embedding strategy we will (inductively) have many candidates for each x preceding y, at most a few of which are crossed off for y (or for any other vertex). When we embed x to a vertex which is not crossed off for v, the fraction of unpromising extensions of $H'_1(y)$ can increase, but since this fraction starts tiny and is increased only by a bounded amount a bounded number of times, it will remain small. In particular, we will end up embedding $H'_0(y)$ to a promising subgraph, which guarantees y has many candidates as we needed.

We briefly mention one place in this sketch where significant technical work is required. This is in the construction of G, [1, Lemma 19]. To make this strategy work, the number of unpromising subgraphs we can allow must be very tiny indeed. In particular, a standard application of the Sparse Regularity Lemma [20] will not give sufficiently good control of the constants, and we need instead to use a strengthened version of the Sparse Regularity Lemma together with a 'cleaning' process. In order to obtain the required counting results, we in addition need the 'Counting KŁR' results of Conlon, Gowers, Samotij and Schacht [8].

We should also note that the 'mixing rapidly' proof of [1, Lemma 20] relies on Spencer's theorem [26] on rooted copies.

We now sketch how this strategy can be modified to work for $k \geq 3$. The object G into which we embed F needs to be a k-complex (hypergraph with edges of size at most k) with its edges of size k selected from edges of the same colour in Γ. In order to show that a suitable G exists, we need to use a strengthened sparse version of the Strong Hypergraph Regularity Lemma, proved by Allen, Parczyk and Pfenninger [4]. We need to develop a hypergraph version of the Counting KŁR results of [8] for this setting of complexes. And, finally, we need to use a rather more involved 'cleaning' process in order that we obtain the required control of our constants for 'few unpromising subgraphs'.

Apart from this, much of our strategy sketched above works in a broadly similar way for hypergraphs. We need to use the polynomial concentration theorem of Kim and Vu [19] replacing Spencer's theorem. We need to view F as a complex by down-closure and hence we need to consider edges of all uniformities up to k, not just of uniformity k, throughout. In particular, although all edges of F of uniformity smaller than k are contained in edges of uniformity k, this property is not preserved for the subgraphs $H'_0(y)$ and $H'_1(y)$; these k-complexes can have edges of smaller uniformity that are not in any k-edge.

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