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On the number of tangencies among 1-intersecting curves

(Extended abstract)

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Abstract

Let $\mathcal{C}$ be a set of curves in the plane such that no three curves in $\mathcal{C}$ intersect at a single point and every pair of curves in $\mathcal{C}$ intersect at exactly one point which is either a crossing or a touching point. According to a conjecture of János Pach the number of pairs of curves in $\mathcal{C}$ that touch each other is $O(|\mathcal{C}|)$. We prove this conjecture for x-monotone curves.

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1 Introduction

We study the number of tangencies within a family of 1-intersecting x-monotone planar curves. A planar curve is a *Jordan arc*, that is, the image of an injective continuous function from a closed interval into $\mathbb{R}^2$. If no two points on a curve have the same x-coordinate, then the curve is *x-monotone*. We consider families of curves such that every pair of curves

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intersect at a finite number of points. Such a family is called t-intersecting if every pair of curves intersects at at most t points. An intersection point p of two curves is a *crossing point* if there is a small disk D centered at p which contains no other intersection point of these curves, each curve intersects the boundary of D at exactly two points and in the cyclic order of these four points no two consecutive points belong to the same curve. If two curves intersect at exactly one point which is not a crossing point, then we say that they are *touching* or *tangent* at that point.

The number of *tangencies* is the number of tangent pairs of curves. If more than two curves are allowed to intersect at a common point, then every pair of curves might be tangent, e.g., for the graphs of the functions x^{2i}, $i = 1, 2, \ldots, n$, in the interval $[-1, 1]$. Therefore, we restrict our attention to families of curves in which no three curves intersect at a common point. It is not hard to construct such a family of n (x-monotone) 1-intersecting curves with $\Omega(n^{4/3})$ tangencies based on a famous construction of Erdős (see [14]) of n lines and n points admitting that many point-line incidences. János Pach [13] conjectured that requiring every pair of curves to intersect (either at crossing or a tangency point) leads to significantly less tangencies.

Conjecture 1 ([13]). *Let $\mathcal{C}$ be a set of n curves such that no three curves in $\mathcal{C}$ intersect at a single point and every pair of curves in $\mathcal{C}$ intersect at exactly one point which is either a crossing or a tangency point. Then the number tangencies among the curves in $\mathcal{C}$ is $O(n)$.*

Györgyi, Hujter and Kisfaludi-Bak [8] proved Conjecture 1 for the special case where there are constantly many faces in the arrangement of $\mathcal{C}$ that together contain all the endpoints of the curves. In this paper we show that Conjecture 1 also holds for x-monotone curves.

Theorem 2. *Let $\mathcal{C}$ be a set of n x-monotone curves such that no three curves in $\mathcal{C}$ intersect at a single point and every pair of curves in $\mathcal{C}$ intersect at exactly one point which is either a crossing or a tangency point. Then the number tangencies among the curves in $\mathcal{C}$ is $O(n)$.*

We prove Theorem 2 by considering two types of tangencies according to whether a tangency point is between two curves such that their projections on the x-axis are nested or overlapping. In each case we consider the tangencies graph whose vertices represent the curves and whose edges represent tangent pairs of curves. In the latter case we show that it is possible to disregard some ratio of the edges using the pigeonhole principle and the dual of Dilworth's Theorem and then order the remaining edges such that there is no long monotone increasing path with respect to this order. In the first case, we show that after disregarding some ratio of the edges the remaining edges induce a forest. Due to space limitations most of the details of the proof are omitted. The interested reader can find them in [3].

Related Work. It follows from a result of Pach and Sharir [17] that n x-monotone 1-intersecting curves admit $O\left(n^{4/3} (\log n)^{2/3}\right)$ tangencies. Note that this bound almost matches the lower bound mentioned above. It also follows from [17] that for bi-infinite x-monotone 1-intersecting curves the maximum number of tangencies is $\Theta(n \log n)$. Pálvölgyi et nos [2] showed that there are $O(n)$ tangencies among families of n 1-intersecting curves that can be partitioned into two sets such that all the curves within each set are pairwise disjoint. Variations of this bipartite setting were also studied in [1, 10, 19].

Pach, Rubin and Tardos [15, 16] settled a long-standing conjecture of Richter and Thomassen [20] concerning the number of crossing points determined by pairwise intersecting curves. In particular, they showed that in any set of curves admitting linearly many tangencies the number of crossing points is superlinear with respect to the number of tangencies. This implies that for any fixed t every set of n t-intersecting curves admits $o(n^2)$ tangencies. Salazar [22] already pointed that out for such families which are also pairwise intersecting. Better bounds for families of t-intersecting curves were found in [5, 10]. Specifically, it follows from [10] that n 1-intersecting curves determine $O(n^{7/4})$ tangencies.

There are several other problems in combinatorial geometry that can be phrased in terms of bounding the number of tangencies between certain curves, see, e.g., [4]. The most famous of which is the unit distance problem of Erdős [6] which asks for the maximum number of unit distances among n points in the plane. It is easy to see that this problem is equivalent to asking for the maximum number of tangencies among n unit circles.

2 Proof of Theorem 2

Let $\mathcal{C}$ be a set of n x-monotone curves such that no three curves in $\mathcal{C}$ intersect at a single point and every pair of curves in $\mathcal{C}$ intersect at exactly one point which is either a crossing or a tangency point. By slightly extending the curves if needed, we may assume that every intersection point of two curves is an interior point of both of them and that all the endpoints of the curves are distinct.

Let $p = (x_1, y_1)$ and $q = (x_2, y_2)$ be two points. We write $p <_x q$ if $x_1 < x_2$ and we write $p <_y q$ if $y_1 < y_2$. We mainly consider the order of points from left to right, so when we use terms like 'before', 'after' and 'between' they should be understood in this sense. For a curve $c \in \mathcal{C}$ we denote by $L(c)$ and $R(c)$ the left and right endpoints of c, respectively. If $p, q \in c$, then $c(p, q)$ denotes the part of c between these two points. We denote by $c(-, p)$ (resp., $c(p, +)$) the part of c between $L(c)$ (resp., $R(c)$) and p. For another curve $c' \in \mathcal{C}$ we denote by $I(c, c')$ the intersection point of c and c'. We may also write, e.g., $c(c', q)$ instead of $c(I(c, c'), q)$

Suppose that an x-monotone curve c_1 lies *above* another x-monotone curve c_2, that is, the two curves are non-crossing (but might be touching) and there is no vertical line ℓ such that $I(c_1, \ell) <_y I(c_2, \ell)$. Assuming the endpoints of c_1 and c_2 are distinct there are four possible cases: (1) $L(c_1) <_x L(c_2) <_x R(c_2) <_x R(c_1)$; (2) $L(c_2) <_x L(c_1) <_x R(c_1) <_x R(c_2)$; (3) $L(c_1) <_x L(c_2) <_x R(c_1) <_x R(c_2)$; and (4) $L(c_2) <_x L(c_1) <_x R(c_2) <_x R(c_1)$. We denote by $c_2 \prec_i c_1$ the relation that corresponds to case i, for $i = 1, 2, 3, 4$. It is not hard to see that each $\prec_i$ is a partial order.

Proposition 3. *For every $i = 1, 2, 3, 4$ there are no three curves $c_1, c_2, c_3 \in \mathcal{C}$ such that $c_1 \prec_i c_2 \prec_i c_3$.*

We say that the tangency point of two touching curves $c_1, c_2 \in \mathcal{C}$ is of *Type i* if $c_1 \prec_i c_2$. We will count separately tangency points of Types 1 and 2 and tangency points of Types 3 and 4.

Lemma 4. *There are $O(n)$ tangency points of Type 1 or 2.*

Proof. Since all the curves in $\mathcal{C}$ are pairwise intersecting and x-monotone there is a vertical line ℓ that intersects all of them. By slightly shifting ℓ if needed we may assume that no two curves intersect ℓ at the same point. We assume without loss of generality that at least half of all the tangency points of Types 1 and 2 are to the right of ℓ, for otherwise we may reflect all the curves about ℓ. We may further assume that at least half of the tangency points of Types 1 and 2 to the right of ℓ are of Type 2, for otherwise we may reflect all the curves about the x-axis. Henceforth, we consider only Type 2 tangency points to the right of ℓ.

By Proposition 3 a curve cannot touch one curve from above and another curve from below at Type 2 tangency points. Thus, we may partition the curves into *blue* curves and *red* curves such that at every tangency point a blue curve touches a red curve from below (we ignore curves that contain no tangency points among the ones that we consider).

Proposition 5. *Every pair of blue curves cross each other.*

We proceed by marking the rightmost tangency point on every red curve. Clearly, at most n tangency points are marked. Henceforth, we consider only unmarked tangency points. Let G be the (bipartite) *tangencies graph* of the blue and red curves. That is, the vertices of G correspond to the blue and red curves and its edges correspond to pairs of touching blue and red curves (recall that we consider only unmarked tangency points of Type 2 to the right of ℓ). We will show that G is a forest and hence has at most $n - 1$ edges.

Suppose that G contains a cycle and let $C = b_0 - r_0 - b_1 - r_1 - \ldots - b_k - r_k - b_0$ be a shortest cycle in G, such that b_i corresponds to a blue curve and r_i corresponds to a red curve, for every $i = 0, 1, \ldots, k$. We may assume without loss of generality that b_1 has the lowest intersection point with ℓ among the blue curves in C and that $I(b_0, \ell) <_y I(b_2, \ell)$.

Proposition 6. *For every $i \geq 1$ the curve r_i intersects ℓ above r_0 and intersects $b_0(-, \ell)$, $r_0(b_0, +)$ and $b_1(b_0, +)$. See Figure 1 for an illustration.*

It follows from Proposition 6 that r_k intersects b_0 to the left of ℓ and therefore (b_0, r_k) cannot be an edge in G. Thus G is a forest and has at most $n - 1$ edges. This implies that there are at most $2n - 1$ Type 2 tangency points to the right of ℓ and at most $8n - 4$ tangency points of Types 1 and 2. □

Lemma 7. *There are $O(n)$ tangency points of Type 3 or 4.*

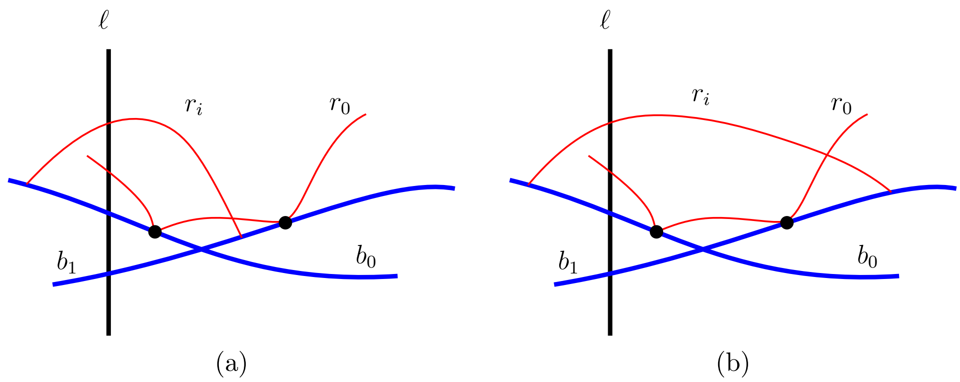

Figure 1: Illustrations for the statement of Proposition 6: r_i intersects ℓ above r_0 and intersects $b_0(-,\ell)$, $r_0(b_0,+)$ and $b_1(b_0,+)$.

Proof. As in the proof of Lemma 4, we may assume that there is a vertical line ℓ that intersects all the curves at distinct points and it is enough to consider only Type 4 tangency points to the right of ℓ.

By Proposition 3 a curve cannot touch one curve from above and another curve from below at Type 4 tangency points. Thus, we may partition the curves into *blue* curves and *red* curves such that at every tangency point a blue curve touches a red curve from below (we ignore curves that contain no tangency points among the ones that we consider).

Clearly, there are no Type 4 tangencies among the blue curves, however, there might be tangencies of other types among them. Next we wish to obtain a subset of the blue curves such that every pair of them are crossing and they together contain a percentage of the tangency points that we consider. It follows from Proposition 3 that the largest chain in the partially ordered set of the blue curves with respect to $\prec_1$ is of length two. Therefore, by Mirsky's Theorem (the dual of Dilworth's Theorem) the blue curves can be partitioned into two antichains with respect to $\prec_1$. The blue curves of one of these antichains contain at least half of the tangency points that we consider. By continuing with this set of blue curves and applying the same argument twice more with respect to $\prec_2$ and $\prec_3$ we obtain a set of pairwise crossing blue curves that together contain at least $1/8$ of the tangency points of Type 4 to the right of ℓ. Henceforth we consider these blue curves and the red curves that touch at least one of them at a Type 4 tangency point to the right of ℓ.

Let $G=(B\cup R,E)$ be the (bipartite) *tangencies graph* of these blue and red curves. That is, B corresponds to the blue curves, R corresponds to the red curves and E corresponds to pairs of touching blue and red curves (at Type 4 tangency points to the right of ℓ). We order the edges of G according to the order of their corresponding tangency points from left to right. We will show that G has linearly many edges using the following fact, attributed to Rödl [21] in [7].

Proposition 8. *Let $G=(V,E)$ be a graph and let $<$ be a total order of its edges. Let k be an integer and suppose that G does not contain a monotone increasing path of k edges, that is, a path $e_1-e_2-\ldots-e_k$ such that $e_1<e_2<\ldots<e_k$. Then $|E|<\binom{k}{2}|V|$.*

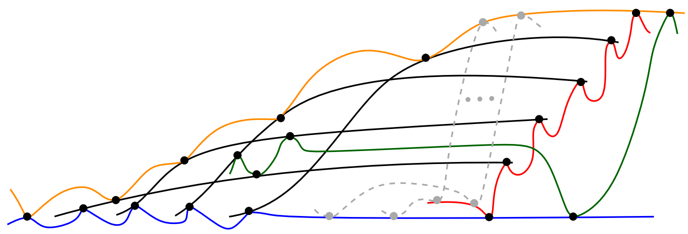

Figure 2: n x-monotone pairwise intersecting 1-intersecting curves might determine $3n-4$ tangencies.

Recall that we order the edges of G according to the order of their corresponding tangency points from left to right. The lemma follows from Proposition 8 and the next claim.

Proposition 9. *G does not contain a monotone increasing path of 7 edges starting at B.*

We conclude from Propositions 8 and 9 that G has at most $28n$ edges. This in turn implies that there are at most $8 \cdot 2 \cdot 2 \cdot 28n = 896n$ tangency points of Types 3 and 4. $\square$

By Lemmata 4 and 7 there are at most $904n - 4$ tangency points among the curves in $\mathcal{C}$. This concludes the proof of Theorem 2.

3 Discussion

We have shown that n x-monotone pairwise intersecting 1-intersecting curves determine $O(n)$ tangencies. The constant hiding in the big-O notation is rather large, since, for simplicity, we did not make much of an effort to get a smaller constant. In particular, our upper bound can be improved by considering more cases. It would be interesting to determine the exact maximum number of tangencies among a set of n x-monotone curves each two of which intersect at exactly one point. The best lower bound we came up with is $3n-4$, see Figure 2.

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